Online Appendix for "A Theory of Dynamic Contracting with Financial Constraints"

Claim 2. $E = \{\mathbf{w} \in W : w_H \geq \underline{w}^e_H$ *and* $w_L \geq w_H + \kappa\}$.

Proof. Relax the constraints set in $\boxed{\star}$ ignoring the low cost cash-strapped constraint and let $\tilde{E}$ be the set of $\mathbf{w} \in W$ such that this constraints set is non-empty. As before, we construct the sequence of sets $\tilde{E}_l$ with $\tilde{E}_l \subseteq \tilde{E}_{l-1}$ for any l and $\tilde{E} \subseteq \bigcap_{l=0}^{+\infty} \tilde{E}_l = \{\mathbf{w} \in W : w_H \geq \underline{w}^e_H$ and $w_L \geq w_H + \kappa\}$.

Fix $a \in [0, \kappa)$ and $b \in [0, w^*_H)$, then let $\tilde{E}^{old}_{a,b} = \{\mathbf{w} \in W : w_H \geq b$ and $w_L \geq w_H + a\}$ and define $\tilde{E}^{new}_{a,b} = \{\mathbf{w} \in W : \exists(\mathbf{w}_H, \mathbf{w}_L) \in \tilde{E}^{old}_{a,b} \times \tilde{E}^{old}_{a,b}$ s.t. (1)$\}$. It follows that $\tilde{E}^{new}_{a,b} = \{\mathbf{w} \in W : w_H \geq \delta(b + \alpha_H a)$ and $w_L \geq w_H + \Delta\theta q^e(\theta_H) + \delta(\alpha_L - \alpha_H)a\}$.

So, define $a_0 = 0$, $b_0 = 0$ and $a_l = \Delta\theta q^* e(\theta_H) + \delta\theta(\alpha_L - \alpha_H)a_{l-1}$, $b_l = \delta(b_{l-1} + \alpha_H a_{l-1})$. Finally, set $\tilde{E}_l = \tilde{E}^{old}_{a_l,b_l}$. The claim follows from $a_{l-1} < a_l < \kappa$, $b_{l-1} < b_l < w^e_H$ for any l with $a_l \rightarrow_{l\rightarrow\infty} \kappa$, $b_L \rightarrow_{l\rightarrow\infty} \underline{w}^e_H$. □

Claim 4.

1. *Each* Q^*_j *is concave.*
2. *Each* Q^*_j *is supermodular.*
3. *Each* Q^*_j *is continuously differentiable on* $int(W)$ *with*

$$\lim_{w_L \rightarrow w_H} D_L Q^*_j(\mathbf{w}) = \infty \; \forall \; w_H \text{ and } \lim_{w_H \rightarrow 0} D_H Q^*_j(\mathbf{w}) = \infty \; \forall \; w_L \neq 0$$

4. *Each* Q^*_j *is strictly concave in* w_L *and* w_H *on*

$$H = \{\mathbf{w} \in int(W) : DQ^*_L(\mathbf{w}) \gg 0 \text{ and } DQ^*_H(\mathbf{w}) \gg 0\}$$

Proof.

Part 1. The argument is standard, we need to show that the Bellman operator, defined in $(\mathcal{RF})$, preserves concavity. Indeed, the constraint set is convex and $s(\theta, q)$ is concave in q. So, concavity is preserved by the Bellman operator.

Since the set of concave functions is closed in the space of continuous bounded functions, the result follows from Theorem 3.1 and its Corollary 1 of Stokey et al. [1989].

Part 2. Again, we show that the Bellman operator preserves supermodularity. Attach slack variables v and u_L, u_H to the constraints in $(\mathcal{RF})$ and substitute the third constraint into the first one. In the matrix notations the constraints could be written as:

$$\underbrace{\begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}}_{B} \begin{pmatrix} w_L \\ w_H \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 0 & \delta\alpha_L & \delta(1-\alpha_L) & \Delta\theta & 1 & 0 & 1 \\ \delta\alpha_L & \delta(1-\alpha_L) & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & \delta\alpha_H & \delta(1-\alpha_H) & 0 & 0 & 0 & 1 \end{pmatrix}}_{A} \begin{pmatrix} z_{LL} \\ z_{LH} \\ z_{HL} \\ z_{HH} \\ q_H \\ \nu \\ u_L \\ u_H \end{pmatrix}$$

Of course, also $\mathbf{z}_H, \mathbf{z}_L \in W$, $q_H \in \mathbb{R}_+$ and the slack variables must be non-negative, which gives a closed convex sublattice. We ignored q_L, because its value does not affect any of the constraints.

Call the matrix on the left B and the matrix on the right A. Then, A and B satisfy the conditions in Remark 1 of Theorem 1 of Chen et al. [2013]. These conditions are as follows: $B'B$ has positive diagonal elements, non-positive off-diagonal elements and $B'A \geq 0$. So, supermodularity is preserved by the Bellman operator.

Since the set of supermodular functions is closed in the space of continuous bounded functions, the result follows from Theorem 3.1 and its Corollary 1 of Stokey et al. [1989].

Part 3. Unfortunately, the standard argument of Benveniste and Scheinkman [1979] is not applicable in our context, because it might not to be possible to change q_H keeping $\mathbf{z}_L$, $\mathbf{z}_H$ constant. Moreover, the other known result of Rincón-Zapatero and Santos [2009] also does not have a bite.

We approach the differentiability through the uniqueness of Lagrange multipliers by applying Theorem 2 of Morand and Reffett [2015].

Consider $\boxed{\star}$ and notice that the constraints set in this problem could be described by a linear operator from l^∞ to itself. We shall call this operator the constraint map. First, we argue that the strict Slater's condition is satisfied on $int(W)$. In other words, $\forall \mathbf{w} \in int(W)$ there exists a feasible point such that the constraint map is uniformly bounded away from 0.

Our argument is constructive, consider the cone $Z \subseteq \mathbb{R}^2_+$ defined by

$$z_L \geq \delta[\alpha_L z_L + (1-\alpha_L) z_H]$$
$$z_H \geq \delta[\alpha_H z_L + (1-\alpha_H) z_H]$$

Since this cone has a non-empty interior, there is q_H and a point in it satisfying $z_L > z_H$ and

$$w_L - w_H > \Delta\theta q_H + \delta(\alpha_L - \alpha_H)(z_L - z_H)$$
$$z_L - z_H > \Delta\theta q_H + \delta(\alpha_L - \alpha_H)(z_L - z_H)$$
$$w_L > \delta[\alpha_L z_L + (1-\alpha_L) z_H]$$
$$w_H > \delta[\alpha_H z_L + (1-\alpha_H) z_H]$$

Set $U(\theta_i|h^{t+s+1}) = z_i$ for $i = L, H$ and $\forall h^{t+s+1} \in H|_{h^t}$ and $q(\theta_H|h^{t+s}) = q_H$ $\forall h^{t+s} \in H|_{h^t}$, $\forall s$. This condition guarantees that the solution could be characterized with the Lagrangian approach with multipliers in l^1.

Only the incentive and cash-strapped constraints could bind for the solution $\langle \mathbf{U}^*, \mathbf{q}^* \rangle$ with $\mathbf{w} \in int(W)$. To see this, consider any $h^{t+s} \in H|_{h^t}$ for some s. Since $D_q s(\theta_H, q) \to_{q\to 0} +\infty$, $q^*(\theta_H|h^{t+s}) > 0$. Then, $U^*(\theta_L|h^{t+s}) > U^*(\theta_H|h^{t+s}) \geq 0$ by $IC_L(h^{t+s})$. And, $C_H(h^{t+s})$ with $U^*(\theta_L|h^{t+s}, \theta_H) > U^*(\theta_H|h^{t+s}, \theta_H)$ imply that $U^*(\theta_H|h^{t+s}) > 0$.

Consider the operator mapping l^∞ to itself which sends $\langle \mathbf{U}, \mathbf{q} \rangle$ to the slack variables $\langle \nu, \mathbf{u_L}, \mathbf{u_H} \rangle$. This operator is clearly surjective. Therefore, by Theorem 3 of Morand and Reffett [2015], the Lagrange multipliers are unique. Continuous differentiability follows from uniqueness of these multipliers and concavity of the value function.

Now, we show that $\lim_{w_L \to w_H} D_L Q_j^*(w) = +\infty$. First of all, $D_L Q_j^*(\mathbf{w})$ is continuous, non-increasing in w_L by concavity, therefore $\forall w_H$, $\lim_{w_L \to w_H} D_L Q_j^*(w)$ exists in the extended real line. For w_L close to w_H,

$$Q_j^*(\mathbf{w}) - Q_j^*(w_H, w_H) \geq (1 - \alpha_j) s(\theta_H, (w_L - w_H)/\Delta\theta)$$

because the solution at (w_H, w_H) is also feasible at $\mathbf{w}$. Similarly, $\lim_{w_H \to 0} D_H Q_j^*(\mathbf{w}) = \infty \ \forall \ w_L \neq 0$, because $q_H(z_H) = 0$ when $w_H = 0$.

Part 4. Consider the problem $\boxed{\star}$ and take $\alpha \in (0, 1)$. Let $\langle \mathbf{U}^*, \mathbf{q}^* \rangle$ and $\left\langle \mathbf{U}^{*'}, \mathbf{q}^{*'} \right\rangle$ be the solutions to the problem at $\mathbf{w} \in H$ and $(w'_L, w_H) \in H$ with $w_L > w'_L$, respectively. By the definition of H, $Q_j^*(\mathbf{w}) > Q_j^*(w'_L, w_H)$ which implies $\mathbf{q}^* \neq \mathbf{q}^{*'}$. Notice that $(\alpha \mathbf{U}^* + (1 - \alpha)\mathbf{U}^{*'}, \alpha \mathbf{q}^* + (1-\alpha)\mathbf{q}^{*'})$ is feasible at $\alpha \mathbf{w} + (1-\alpha)(w'_L, w_H)$ and it strictly improves the objective. The similar argument establishes concavity in the other coordinate. □

Lemma . *β is non-increasing in w_L, non-decreasing in w_H.*

Proof. To prove the former part, assume its converse, namely for some $w_L > w'_L > w_H > 0$, $\beta(\mathbf{w}) > \beta(w'_L, w_H)$. By Equation the first-order condition, $D_q s(\theta_H, q_H) = \Delta\theta\beta$, $q_H(\mathbf{w}) < q_H(w'_L, w_H)$. It follows from the incentive constraint that $z_{HL}(\mathbf{w}) - z_{HH}(\mathbf{w}) > z_{HL}(w'_L, w_H) - z_{HH}(w'_L, w_H)$. Hence, the cash-strapped constraint implies that $z_{HL}(w'_L, w_H) < z_{HL}(\mathbf{w})$, but $z_{HH}(w'_L, w_H) > z_{HH}(\mathbf{w})$. Therefore, $D_L Q_H^*[z_H(w'_L, w_H)] \geq D_L Q_H^*[z_H(\mathbf{w})]$, on the other hand $D_H Q_H^*[z_H(w'_L, w_L)] \leq D_H Q_H^*[z_H(\mathbf{w})]$ by concavity and supermodularity. Invoking the remaining first-order conditions and $\beta(\mathbf{w}) > \beta(w_L, w'_H)$, $\rho_H(\mathbf{w}) > \rho_H(w'_L, w_H) > \rho_H(\mathbf{w})$, which is impossible.

The latter part immediatelly follows from the fact that ρ_L is independent of w_H, supermodularity and from the envelope conditions: $D_L Q_L^*(\mathbf{w}) = \alpha_L \rho_L + (1 - \alpha_L)\beta$.

□

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