

# Reduced Forms: Feasibility, Extremality, Optimality

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## Abstract

We study independent private values auction environments in which the auctioneer’s revenue depends nonlinearly on bidders’ interim winning probabilities. Our framework accommodates heterogeneity among bidders and places no ad hoc constraints on the mechanisms available to the auctioneer. Within this general setting, we show that feasibility of interim winning probabilities can be tested along a unidimensional curve—the *principal curve*—and use this insight to explicitly characterize the extreme points of the feasible set. We then combine our results on feasibility and extremality to solve for the optimal auction under a natural regularity condition. We show that the optimal mechanism allocates the good based on *principal virtual values*, which extend Myerson’s virtual values to nonlinear settings and are constructed to equalize bidders’ marginal revenue along the principal curve. We apply our approach to the classical linear model, settings with endogenous valuations due to ex ante investments, and settings with non-expected utility preferences, where previous results were largely limited either to symmetric environments with symmetric allocations or to two-bidder environments.

## I Introduction

Auctions are among the most widely used market institutions. Governments use them to allocate spectrum licenses and procurement contracts; firms employ them to acquire or liquidate inputs; and millions of individuals participate daily on online platforms. A central question in auction theory is how to design mechanisms that maximize an auctioneer’s objective, such as expected revenue.

The benchmark case of independent private values (IPV) with linear utilities is well understood following Myerson [1981]. Myerson’s celebrated analysis shows that in this environment the optimal mechanism awards the good to the bidder with the highest nonnegative virtual value under mild regularity conditions. A key feature that makes this analysis tractable is that expected revenue is linear in bidders’ interim winning probabilities. This linearity allows the auctioneer’s problem to be solved pointwise over feasible allocations in regular environments when ironing is not needed.

Much less is known once the auctioneer’s objective departs from linearity. Nonlinear dependence on interim winning probabilities arises naturally in economically important settings—for instance, when bidders undertake ex ante investments that affect valuations, as in models of endogenous valuations (Gershkov et al. [2021]), or when bidders have non-expected-utility preferences such as constant relative risk aversion (CRA) (Gershkov et al. [2022]). In such environments, because the auctioneer’s revenue depends nonlinearly on interim winning probabilities, the convenient pointwise maximization of the linear benchmark is no longer available. As a result, classical methods break down, and the analysis of optimal auctions becomes substantially more complex.

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This paper develops a unified framework for characterizing optimal auctions when the auctioneer’s expected revenue can be written in reduced form as an integral of bidder-specific revenue terms that may be nonlinear in interim winning probabilities. We allow for full asymmetry across bidders and do not impose any ad hoc restrictions on the mechanisms available to the auctioneer. Our framework accommodates a wide range of economic environments, including the linear IPV model, settings with endogenous valuations, and settings with CRA preferences cited above.

The core challenge in nonlinear environments is the loss of separability: the auctioneer’s problem no longer decomposes across types, and allocations cannot be chosen type-by-type. Making progress therefore requires resolving three intertwined challenges.

*Feasibility.* Bidders’ interim winning probabilities must be consistent with some ex post allocation, and characterizing this consistency requires checking Border’s condition. This condition provides a set of necessary and sufficient inequalities, but this set is high-dimensional—one inequality for every possible cutoff vector of types. While tractable for symmetric auctions or with two bidders, it quickly becomes unmanageable as we move to asymmetric auctions with a larger number of bidders. This leaves little hope of solving the fully constrained problem directly.

*Extremality.* The set of feasible reduced forms—tuples of bidders’ interim winning probabilities—is convex. Consequently, when the auctioneer’s objective is convex—as in many endogenous-valuation environments—an optimum is attained at an extreme point of the feasible set. A tractable description of extremal reduced forms would therefore sharply restrict the set of candidates and reduce the subsequent maximization problem to a much lower-dimensional search. Yet, existing characterizations are largely confined to symmetric settings or to two bidders, leaving the structure of extreme points poorly understood in richer settings.

*Optimality.* Ultimately, the core difficulty is that nonlinearity destroys the pointwise structure of the linear benchmark. Feasibility and extremality help identify the relevant constraint set and restrict attention to plausible candidates, but the remaining task is still to optimize a nonlinear objective over that set. Doing so requires a methodology that both solves the auctioneer’s problem and delivers sharp structural insights about the optimal auction.

In this paper, we address these challenges by developing new characterizations for feasibility and extremality and then leveraging them to solve the auctioneer’s problem.

Our first contribution is a new representation of feasibility that collapses Border’s high-dimensional family of constraints to a unidimensional test. We show that for any reduced form there exists a canonical *principal curve* inside the unit cube with the property that Border’s inequalities hold everywhere if and only if they hold along this single curve. This reduction makes feasibility straightforward to verify even in fully asymmetric environments with many bidders.

Our second contribution is a sharp characterization of extremality. Building on our feasibility representation, we show that a reduced form is extremal precisely when Border’s inequalities bind along its principal curve. This criterion dramatically simplifies the auctioneer’s problem in environments where the optimum is known to be extremal. It also clarifies how to induce reduced forms through ex post allocations. Specifically, we show that every extremal reduced form can be implemented by a simple *score allocation*. In a score allocation, each bidder’s type is mapped into a unidimensional score; the good is awarded to the bidder with the highest nonnegative score; and ties occur with probability zero. We prove that score allocations are exactly the mechanisms that generate extremal reduced forms. This yields a sharper form of the classical equivalence between Bayesian and dominant-strategy implementation originally established by [Manelli and Vincent \[2010\]](#). In particular, we establish that any reduced form is implementable as a mixture of deterministic, dominant-strategy incentive compatible, and nonbossy allocations, in the sense of [Satterthwaite and Sonnenschein \[1981\]](#).

Our third contribution concerns optimality in nonlinear environments. We reformulate the auctioneer’s prob-

lem as a continuous-time optimal control problem that builds the principal curve over “time”, moving from high to low types. In this representation, the auctioneer chooses how quickly to allocate winning probability to each bidder, subject to a dynamic feasibility constraint that captures Border’s condition along the curve. This reformulation opens the door to standard tools from control theory and yields structural and economic insights that were unavailable before. Specifically, the optimality conditions reveal a key organizing principle: on segments where monotonicity does not bind (i.e., where no ironing is required), the optimal principal curve equalizes bidders’ marginal revenues. Using this insight, we completely solve for the optimum in regular environments where ironing is not required.

In regular environments where the optimum is extremal, marginal revenue equalization delivers a complete characterization of the optimal auction, which takes the form of a score allocation. Each bidder’s score is a *principal virtual value*, constructed to equalize bidders’ marginal revenues along the optimal principal curve. In linear settings, principal virtual values coincide with Myerson’s virtual values since bidders’ marginal revenues do not depend on their interim winning probabilities. In nonlinear environments, these scores depart from Myerson’s virtual values because they incorporate the shadow cost of feasibility: increasing one bidder’s interim winning probability necessarily comes at the expense of others and therefore changes their marginal revenues.

We show how this characterization applies to the endogenous-valuation model, where bidders choose ex ante investments that affect their valuations. In this environment, interim winning probabilities matter not only for rent extraction, but also because they determine the return to investing. Any bidder with a positive interim winning probability has an incentive to invest even though only one ultimately wins, so investment costs are largely duplicated. The optimal auction departs from Myerson’s auction by shifting winning probability toward bidders with higher investment returns and away from bidders with lower investment returns, until bidders’ marginal revenues are equalized along the optimal principal curve. Relative to Myerson’s auction, this reallocation reduces excessive duplication of investment costs, thereby increasing the auctioneer’s expected revenue.

We then extend the analysis to regular environments in which the optimum is not extremal. This case arises naturally when the auctioneer’s objective exhibits “satiation” in interim winning probabilities, as in settings with risk-averse bidders. In such environments, the score ranking implied by principal virtual values remains informative about who should receive probability, but it no longer pins down the amount to allocate. In particular, at low types the marginal revenue from awarding the full unit to the top-ranked bidder can be negative. Rather than excluding such types from trade altogether, the auctioneer can allocate a strictly positive fraction, increasing it until marginal revenue reaches zero. We formalize this behavior through fractional score allocations, which preserve the score ranking while allowing the winning fraction to vary with the bidder’s type.

Finally, we illustrate the implications of this characterization in the CRA model, where awarding a fractional unit to risk-averse bidders at the bottom of the type distribution is beneficial for the auctioneer because it lowers informational rents while generating only small efficiency losses. In a CRA application with two groups of bidders—risk-neutral and risk-averse—we derive a sharp comparative-statics insight. The optimal auction favors the risk-averse bidders at low types and the risk-neutral bidders at high types. This echoes Myerson’s logic that the optimal auction should tilt toward ex ante “weak” bidders to intensify competition; however, the key subtlety in the nonlinear CRA model is that who is “weak” depends on bidders’ interim winning probabilities. At high interim winning probabilities, the risk-averse bidders are effectively stronger because they are willing to pay an additional risk premium to secure the good with near certainty. At low interim winning probabilities, this ranking is reversed. Accordingly, the optimal auction should be biased in favor of risk-averse bidders for low allocations (and thus low types) and against risk-averse bidders for high allocations (and thus high types). As we explain, this reversal is a direct implication of marginal revenue equalization along the optimal principal curve.

## 1.1 Relation to literature

A substantial body of literature has studied feasibility, extremality, and optimality of reduced forms. Below, we describe the key existing contributions and relate our results to them.

*Feasibility.* In a classic result, [Border \[1991\]](#), building on [Maskin and Riley \[1984\]](#) and [Matthews \[1984\]](#), characterized the set of feasible symmetric reduced forms—that is, symmetric interim winning probabilities inducible by some ex post allocation—via a unidimensional family of linear inequalities. [Border \[2007\]](#) and [Mierendorff \[2011\]](#) generalized this result to asymmetric reduced forms, where feasibility is described by a family of inequalities whose dimension grows with the number of bidders. [Che et al. \[2013\]](#), in turn, extended these results to multi unit auction settings with capacity constraints. [Hart and Reny \[2015\]](#) related Border’s conditions to second-order stochastic dominance, and [He et al. \[2025\]](#) showed that Border-type inequalities can also characterize the set of feasible belief distributions generated by private-private information structures.

The closest paper to ours regarding feasibility in auction settings is [Cai et al. \[2018\]](#), who showed that in finite (potentially only partially ordered) type spaces one can reduce the family of asymmetric Border inequalities to one dimension. Our result is complementary to theirs: it relies on additional structure—continuous, unidimensional, linearly ordered types and monotone interim winning probabilities—but delivers a sharper characterization. In this setting, we characterize feasibility via a unidimensional monotone curve, which we describe in closed form through an explicit transformation of reduced forms.

*Extremality.* Since a quasiconvex functional attains its maximum at an extreme point of that set<sup>1</sup>, characterizing extreme points has become an increasingly powerful approach to solving many optimization problems of economic interest at once. In a landmark contribution, [Kleiner et al. \[2021\]](#) characterized extreme points of the set of monotone functions that majorize, or are majorized by, a given monotone function. This yielded, aside from applications to contests, optimal persuasion, and delegation, an immediate description of extreme points of the set of feasible symmetric reduced forms, since the symmetric Border constraint could be viewed as a majorization constraint.<sup>2</sup>

Extreme points of asymmetric reduced forms in auction settings are still not fully understood. A notable contribution is [Manelli and Vincent \[2010\]](#), who characterized some extremal asymmetric reduced forms—those that are step functions—and used this description to establish the BIC–DIC equivalence. More recently, [Yang and Yang \[2025\]](#) provided an equivalent characterization of extremal asymmetric reduced forms in the two-bidder case. Their approach differs from ours: it characterizes extreme points of the set of rationalizable tuples of monotone functions. While the connection between rationalizability and majorization is transparent with two bidders, it is unclear how to extend it to characterize extremal reduced forms when the number of bidders exceeds two. In contrast, our approach delivers an explicit description of all extreme points irrespective of the number of bidders.

A few other papers used the extreme-points approach to solve problems of economic significance. Among them, [Manelli and Vincent \[2007\]](#) employed it to study multidimensional screening. [Nikzad \[2023\]](#) and [Candogan and Strack \[2023\]](#) characterized extreme points of the same set as in [Kleiner et al. \[2021\]](#) but with additional linear constraints and applied this approach to study optimal information disclosure. [Arieli et al. \[2023\]](#) described extreme points of the set of mean-preserving contractions of a given prior. [Kleiner et al. \[2024\]](#) studied extreme points of the set of fusions, the multidimensional analog of mean-preserving contractions, and [Lahr and Niemeier \[2024\]](#) studied extreme points in multidimensional monopolistic screening settings. Finally, [Yang and Zentefis \[2024\]](#) characterized extreme points of the set of monotone functions bounded by two given monotone functions and applied this to political economy, persuasion, the psychology of judgment, and security design.

*Optimality.* As noted above, once the objective is nonlinear in interim winning probabilities, the optimization

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<sup>1</sup>This holds under compactness and continuity by a generalization of Bauer’s maximum principle to quasiconvex functionals; see [Ball \[2023\]](#).

<sup>2</sup>[Kleiner et al. \[2021\]](#) also covered multiunit symmetric environments.

problem becomes hard because pointwise optimization is no longer available. Despite these difficulties, several papers have made substantial progress. [Gershkov et al. \[2021\]](#) leveraged the Fan–Lorentz inequality to solve for the optimal symmetric allocation with many goods when valuations are endogenous and provided conditions under which the globally optimal reduced form is symmetric in two-bidder environments; [Andreyanov et al. \[2024\]](#) provided analysis for a related setting with contractible actions. [Zhang \[2017\]](#) characterized the optimum in the symmetric two-bidder endogenous-valuations model with quadratic costs, whereas [Andreyanov et al. \[2026\]](#) solved for the optimum in general symmetric two-bidder settings with endogenous valuations. Finally, [Gershkov et al. \[2022\]](#) characterized the optimal mechanism in symmetric environments with non-expected-utility bidders exhibiting constant relative risk aversion. Our approach covers these applications and unifies and extends the analysis to general asymmetric settings.

## 1.2 Structure of the paper

The paper contains several distinct results, and to help the reader navigate, we indicate key theorems and propositions in brackets throughout this outline. Because the main objects and transformations we introduce are nonstandard, and later arguments rely on this shared language, we recommend reading the paper linearly.

Section 2 introduces the model and explains how several leading auction environments fit into our framework. Section 3 presents an illustrative example that previews the economics of the optimal mechanism.

Sections 5–7 develop the main methodological results. Specifically, Section 5 reduces Border’s high-dimensional feasibility constraints to a unidimensional test along the principal curve (Theorem 1). Section 6 characterizes extremal reduced forms and links them to implementation via score allocations (Theorems 2 and 3). Section 7 reformulates the auctioneer’s problem as an optimal control problem and derives the marginal revenue equalization condition that organizes the analysis of optimal auctions that follows (Proposition 1 and Proposition 2).

Sections 8 and 9 apply these tools to characterize optimal auctions in regular environments, distinguishing between the extremal and non-extremal cases (Theorems 4 and 5).

Section 10 concludes, and the appendix collects proofs and supplementary results.

## 2 Setting

We consider an auction environment with  $n$  bidders, indexed by  $i = 1, \dots, n$ , and a single auctioneer who sells one indivisible unit of a good. Each bidder  $i$  has a private valuation that depends on a unidimensional type  $\theta_i \in \Theta_i \subset \mathbf{R}_+$ , where  $\Theta_i$  is a compact interval. The type  $\theta_i$  is drawn from a distribution  $F_i$  on  $\Theta_i$  that admits a strictly positive and continuous density denoted by  $f_i$ . Types are independently distributed across bidders, although the marginal distributions may differ. We use boldface for vectors of individual variables, e.g.,  $\boldsymbol{\theta} = (\theta_1, \dots, \theta_n)$  stands for a type profile.

Let  $\boldsymbol{\Theta} = \Theta_1 \times \dots \times \Theta_n$  denote the set of all possible type profiles. An **allocation** is a function  $\mathbf{z} = (z_1, \dots, z_n) : \boldsymbol{\Theta} \rightarrow [0, 1]^n$  satisfying

$$\sum_{i=1}^n z_i(\boldsymbol{\theta}) \leq 1 \tag{1}$$

for all type profiles  $\boldsymbol{\theta} \in \boldsymbol{\Theta}$ . Here,  $z_i(\boldsymbol{\theta})$  represents the probability that bidder  $i$  receives the good when the type profile is  $\boldsymbol{\theta}$ , and the constraint simply means that the auctioneer can sell at most one unit. For bidder  $i$ , the **interim**

**winning probability** given their type  $\theta_i$  is defined by

$$x_i(\theta_i) = \mathbf{E}[z_i \mid \theta_i], \quad (2)$$

where the expectation is taken with respect to the independent draws of the other bidders' types.

The setting described thus far corresponds to the standard IPV auction model with linear utilities. It is well known that in this setting, the auctioneer's expected revenue can be expressed as the following function of interim winning probabilities:

$$\sum_{i=1}^n \int_{\Theta_i} \left( \theta_i - \frac{1 - F_i(\theta_i)}{f_i(\theta_i)} \right) x_i(\theta_i) dF_i(\theta_i),$$

where the term in parentheses is often referred to as **Myerson's virtual value** (henceforth **MVV**).

In contrast to the standard model, we remain agnostic about bidders' preferences and instead represent the auctioneer's implied revenue in reduced form. Specifically, let  $J_i(x_i, \theta_i)$  denote the revenue obtained from selling the good to type  $\theta_i$  with probability  $x_i$ , so that expected revenue can be written as

$$\sum_{i=1}^n \int_{\Theta_i} J_i(x_i(\theta_i), \theta_i) dF_i(\theta_i). \quad (3)$$

This specification subsumes the standard model studied by Myerson, and it is far more general since  $J_i$  need not even be linear in  $x_i$ . For example, this framework accommodates environments with endogenous valuations and bidders with CRA preferences.

We are interested in finding the revenue-maximizing mechanism when the auctioneer's revenue is given by (3), subject to bidders' interim winning probabilities, defined by (2) being weakly increasing. This monotonicity requirement is the only substantive assumption we impose: the auctioneer can implement any allocation in which bidders' interim winning probabilities are weakly increasing in their types and is restricted to search among such allocations. As we explain below, this condition is satisfied in the standard model as well as in settings with endogenous valuations and non-expected utility CRA preferences. In all these cases, monotonicity is both necessary and sufficient for implementability; this is a direct consequence of bidders' incentive compatibility.

To streamline the exposition, we assume that  $J_i$  is twice continuously differentiable and that  $\frac{\partial}{\partial x} J_i$  is not identically zero. In the rest of this section, we show how our general framework encompasses several important auction environments studied in the literature.

## 2.1 IPV setting with linear utilities

This is the classical and most extensively studied case, already mentioned above. Bidder  $i$ 's utility takes the form  $\theta_i z_i - t_i$ , where  $t_i$  denotes their payment to the auctioneer, the auctioneer seeks to maximize  $\mathbf{E}[\sum_{i=1}^n t_i]$ . It is well known that only allocations in which bidders' interim winning probabilities are monotone can be implemented in Bayesian Nash equilibrium. In this environment, the auctioneer's revenue coincides with (3), with

$$J_i(x_i, \theta_i) = \left( \theta_i - \frac{1 - F_i(\theta_i)}{f_i(\theta_i)} \right) x_i$$

up to a constant. This constant is pinned down by the participation constraint of the lowest types, which binds at zero, so the auctioneer's problem is exactly of the form described above.

The solution has been well understood since Myerson [1981]. In the regular case—where bidders' MVVs are

strictly increasing—the item is allocated to the bidder with the highest nonnegative MVV.<sup>3</sup> To see it, note that due to linearity in interim winning probabilities, we can rewrite the objective in (3) as

$$\sum_{i=1}^n \int_{\Theta_i} J_i(x_i(\theta_i), \theta_i) dF_i(\theta_i) = \mathbf{E} \left[ \sum_{i=1}^n \left( \theta_i - \frac{1 - F_i(\theta_i)}{f_i(\theta_i)} \right) z_i(\boldsymbol{\theta}) \right], \quad (4)$$

which can be easily maximized pointwise in the space of allocations satisfying (1).

## 2.2 IPV setting with CRA preferences

In an important contribution, [Gershkov et al. \[2022\]](#) (henceforth GMSZ-CRA) extend optimal auction theory to environments in which bidders' preferences over interim winning probabilities depart from expected utility.<sup>4</sup> Under constant relative risk aversion, they showed that the auctioneer's revenue can still be expressed as in (3), with

$$J_i(x_i, \theta_i) = \theta_i x_i - \frac{1 - F_i(\theta_i)}{f_i(\theta_i)} g_i(x_i),$$

up to a constant. Here,  $g_i(x_i)$  is the certainty equivalent of a lottery paying \$1 with probability  $x_i$  and \$0 otherwise. If  $g_i'' \equiv 0$ , bidder  $i$  has expected-utility preferences; when  $g_i'' \geq 0$ , she is risk-averse.

Monotonicity of  $x$  remains both necessary and sufficient for implementability. Thus, the auctioneer's problem retains the same structure as before, but it is considerably harder because  $J_i$  is generally nonlinear in  $x_i$ . As a result, one cannot express the auctioneer's expected revenue as a function of the allocation similarly to (4) and maximize it pointwise.

To gain tractability, GMSZ-CRA focused on symmetric settings in which  $J_i$  and  $F_i$  are identical across bidders. Under risk aversion,  $J_i$  is concave in  $x_i$ , which was their main case of interest. They showed that symmetry of the environment, combined with the concavity of  $J_i$ , implies that the optimal allocation is symmetric, and they derived a closed-form characterization of the corresponding interim winning probabilities.<sup>5</sup> Their analysis, however, does not cover asymmetric environments, which we take up in this manuscript.

## 2.3 IPV setting with endogenous valuations

In another influential paper, [Gershkov et al. \[2021\]](#) (henceforth GMSZ-EV) analyzed models with ex ante investments and showed that they naturally generate preferences that are convex in interim winning probabilities. In their framework, before the auction, each bidder may take a private action  $a_i$  that increases their valuation from winning. Specifically, bidder  $i$ 's utility is

$$v_i(a_i, \theta_i) z_i - C_i(a_i) - t_i,$$

where  $C_i$  is a convex cost incurred regardless of the auction outcome.

GMSZ-EV demonstrated that when bidders are expected-utility maximizers, the auctioneer's expected revenue again takes the form in (3), with

$$J_i(x_i, \theta_i) = \omega_i(x_i, \theta_i) - \frac{1 - F_i(\theta_i)}{f_i(\theta_i)} \frac{\partial}{\partial \theta_i} \omega_i(x_i, \theta_i),$$

up to a constant, where

$$\omega_i(x_i, \theta_i) = \max_{a_i} v_i(a_i, \theta_i) x_i - C_i(a_i)$$

<sup>3</sup>More generally, the optimal allocation is determined by ironed MVVs and may involve randomization among tied bidders.

<sup>4</sup>Their standing assumption is quasilinearity in money.

<sup>5</sup>GMSZ-CRA also described an approach to the optimal symmetric auction when bidders are symmetric but not necessarily risk-averse.

is the maximal expected utility of type  $\theta_i$  given their interim winning probability  $x_i$ . Since  $\omega_i$  is a pointwise maximum of functions linear in  $x_i$ , it is convex in  $x_i$ , and typically  $J_i$  inherits this convexity.<sup>6</sup>

Finding the optimal mechanism therefore requires maximizing the convex functional in (3) over monotone interim winning probabilities  $\mathbf{x}$  that are consistent with an allocation, i.e., that satisfy (1) and (2). Very little is known about problems of this form. Notable exceptions include Zhang [2017], who found the optimum in an additively separable example with  $n = 2$ ; GMSZ-EV, who analyzed optimal symmetric allocations (and their global optimality) in symmetric environments; and Andreyanov et al. [2026], who characterized the asymmetric optimum with two bidders.

To the best of our knowledge, no results exist for asymmetric environments with more than two bidders. To address this gap, we develop a novel approach to the study of optimal auctions in general nonlinear settings without imposing symmetry or restricting the auctioneer to symmetric mechanisms.

### 3 Illustrative example

Since our approach requires several steps to develop, it is instructive to preview its implications for the optimal auction and underlying economics in a concrete example before turning to the full analysis.

Consider an endogenous-valuation model in which each bidder  $i$  can take a private action  $a_i$  at cost  $a_i^2$ , which increases own valuation from winning the good to  $v_i(a_i, \theta_i) = 2\sqrt{\theta_i}a_i$  so that the maximal utility of bidder  $i$  is  $\omega_i(x_i, \theta_i) = \theta_i x_i^2$ . Relative to the standard linear model, the key difference is the quadratic dependence of utility on the interim winning probability. This nonlinearity arises endogenously from bidders' optimal ex ante investment decisions. As a result, the auctioneer's revenue function  $J_i$  equals bidder  $i$ 's MVV multiplied by the square of own interim winning probability.

In this environment, Myerson's optimal auction is generally suboptimal. The reason is that such an allocation discriminates against stronger bidders so as to reduce their information rents (e.g., see Carroll and Segal [2019]), thereby inducing excessive duplication of investment costs at the ex ante stage. By reallocating winning probability away from weaker bidders, the auctioneer can mitigate over-investment and increase his expected revenue.

To see this more concretely, suppose that bidders' type distributions satisfy

$$\theta_i - \frac{1 - F_i(\theta_i)}{f_i(\theta_i)} = (F_i(\theta_i))^{1/\beta_i}$$

for some  $\beta_i \leq 1$ , where  $\Theta_i = \left[\frac{1}{1/\beta_i + 1}, 1\right]$ . For example, if there are two bidders with  $\beta_1 = 1$  and  $\beta_2 = \frac{1}{2}$ , then their type distributions are given by  $F_1(\theta_1) = 2\theta_1 - 1$  and  $F_2(\theta_2) = \frac{\sqrt{12\theta_2 - 3} - 1}{2}$ .<sup>7</sup> Here, a larger value of  $\beta_i$  is indicative of a higher bidder  $i$ 's strength in the sense of hazard ratio ordering.<sup>8</sup> Direct calculations reveal that Myerson's auction yields expected revenue of  $\frac{10}{21}$ , whereas trading exclusively with the stronger bidder—bidder 1—gives  $\frac{1}{2}$ .

More generally, we show in Section 8.2 that the optimal auction allocates the good to the bidder with the highest value of  $(F_i(\theta_i))^{1/\beta_i - 1}$ . It can be verified that this mechanism equalizes marginal revenues,  $\frac{\partial}{\partial x_i} J_i = \frac{\partial}{\partial x_j} J_j$ , whenever bidders  $i$  and  $j$  are tied. As a result, the optimal allocation discriminates less against stronger bidders than Myerson's auction. For example, if bidder  $i$  and bidder  $j$  are tied in Myerson's auction, so that  $(F_i(\theta_i))^{1/\beta_i} =$

<sup>6</sup>It can happen even though  $J_i$  is nonconvex in  $x_i$ , its positive part  $\max\{0, J_i\}$  is convex. As explained in GMSZ-EV, the difference is inconsequential for the structure of the optimum under suitable regularity.

<sup>7</sup>To obtain  $F_i$ , remark that its inverse  $v_i$  satisfies  $v_i(u) - (1-u)v_i'(u) = u^{1/\beta_i}$ . Solving this equation yields  $v_i(u) = \frac{1}{1-u} \int_u^1 s^{1/\beta_i} ds$ .

<sup>8</sup>For two cumulative distribution functions,  $F$  and  $G$ , with the same support,  $F$  is stronger in the sense of hazard ratio ordering than  $G$  if  $\frac{1-F}{f} \geq \frac{1-G}{g}$ . See Section 8.3 for the general definition and discussion of this order.



$(F_j(\theta_j))^{1/\beta_j}$  and  $\beta_i > \beta_j$ , then the stronger bidder  $i$  should win with certainty in the optimal auction.

## 4 Preliminaries and simplifications

As noted above, finding the optimal mechanism is a challenging problem. It is useful to introduce several simplifications and auxiliary observations that streamline the exposition and reduce the dimensionality of the problem.

First, we can absorb all asymmetries across bidders into the auctioneer's revenue function by working with quantiles of bidders' types. Specifically, let  $u_i = F_i(\theta_i)$  denote the quantile of  $\theta_i$ , which is uniformly distributed on  $[0, 1]$ , and define

$$H_i(x_i, u_i) = J_i(x_i, F_i^{-1}(u_i)).$$

Then, our original problem is equivalent to one in which types, denoted by  $\mathbf{u}$ , are uniformly and identically distributed, and the auctioneer's revenue from selling the good to type  $u_i$  with probability  $x_i$  is given by  $H_i(x_i, u_i)$ . In this formulation, both allocations and interim winning probabilities can be viewed as functions of  $\mathbf{u}$ <sup>9</sup>, so that (3) becomes

$$\sum_{i=1}^n \int_0^1 H_i(x_i(u), u) du. \quad (5)$$

Second, observe that the expected revenue in (5) depends only on interim winning probabilities. It has been known since at least Border [1991] and Mierendorff [2011] that  $\mathbf{x}$  is consistent with some allocation  $\mathbf{z}$  satisfying analogues of (1) and (2) in the  $\mathbf{u}$ -space (in which case we say that  $\mathbf{z}$  **induces**  $\mathbf{x}$ ) if and only if the following (Border) constraint holds:

$$B(\mathbf{u}) = \prod_{i=1}^n u_i + \sum_{i=1}^n \int_{u_i}^1 x_i(u) du \leq 1 \quad \forall \mathbf{u} \in [0, 1]^n. \quad (6)$$

This constraint has received considerable attention in the literature. It ensures that interim allocation rules  $\mathbf{x}$  are feasible, in the sense that they can be induced by some ex post allocation satisfying the feasibility condition in (1). Intuitively, (6) requires that for every cutoff vector  $\mathbf{u}$ , the total probability assigned to the bidders whose types lie in the upper tails  $[u_i, 1]$  does not exceed the probability that at least one such bidder is present.

Finally, it is without loss of generality to require each  $x_i$  to be right-continuous with  $x_i(1) = 1$ . To see this, recall that  $x_i$  is weakly increasing and thus has at most countably many jump discontinuities. Redefining  $x_i$  at those points (and possibly at  $u = 1$ ) affects neither the objective in (5) nor the Border constraint in (6).

By this observation, we can restrict attention to  $\mathbf{x}$  that are cumulative distribution functions (CDFs) on the unit interval. Let  $\mathcal{X}$  denote the set of such CDFs. We refer to a tuple  $\mathbf{x} \in \mathcal{X}^n$ , which captures interim winning probabilities, as a **reduced form**. A reduced form  $\mathbf{x}$  is **feasible** if it satisfies the Border constraints in (6), and it is **optimal** if it solves the auctioneer's problem, which can be succinctly restated as

$$\max_{\mathbf{x} \in \mathcal{X}^n} \sum_{i=1}^n \int_0^1 H_i(x_i(u), u) du \quad \text{s.t.} \quad B(\mathbf{u}) \leq 1 \quad \forall \mathbf{u} \in [0, 1]^n. \quad (7)$$

While compact, the maximization problem in (7) remains challenging due to the nonlinearities in  $H_i$ , and the feasible set is defined by a large and intricate family of inequalities. To make progress, we introduce a novel characterization of feasibility that dramatically simplifies Border's condition. Furthermore, since the feasible set is convex,

<sup>9</sup>To obtain allocations in the original type space, we can simply substitute back  $\mathbf{u} = (F_1(\theta_1), \dots, F_n(\theta_n))$ , e.g.,  $x_i(F_i(\theta_i))$  stays for their interim winning probability.

additional tractability can be obtained by analyzing its extreme points.<sup>10</sup> For example, if each revenue function  $H_i$  is convex in its first argument—as often arises in settings with endogenous valuations—then an optimal reduced form can be found at an extreme point of the feasible set. We call a feasible reduced form  $\mathbf{x}$  **extremal** if it is an extreme point of the set of CDFs satisfying the Border constraint in (6).

In the sequel, we characterize all extremal reduced forms and use this characterization, together with our new feasibility result, to solve for the optimum.

## 5 Feasibility

Border’s condition (6) tells us exactly which reduced forms are feasible. The drawback, however, is that it requires checking an entire  $n$ -dimensional family of inequalities—one for every cutoff vector  $\mathbf{u} \in [0, 1]^n$ . When  $n = 2$  this is still manageable, but as soon as there are three or more bidders the condition becomes challenging to verify directly. Our goal in this section is to show that this system of constraints can, in fact, be reduced to a simple unidimensional test.

Our key insight is that feasibility can be checked along a single continuous curve inside the unit cube. That is, for every reduced form  $\mathbf{x}$ , there exists a path

$$s \in [0, 1] \mapsto \boldsymbol{\nu}(s) \in [0, 1]^n$$

such that Border’s inequality holds everywhere if and only if it holds along this path. Instead of monitoring a continuum of constraints in  $n$  dimensions, we can therefore follow just one curve. We refer to this curve  $\boldsymbol{\nu}$  associated with a given reduced form  $\mathbf{x}$  as the **principal curve** of  $\mathbf{x}$ .

### 5.1 $\psi$ -transform

To explicitly derive the principal curve, we need a transformation that takes a cumulative distribution function (CDF)  $x \in \mathcal{X}$  and associates it with a new function  $\psi$ , which we call the  **$\psi$ -transform of  $x$** . The construction works in two steps.

First, consider the map  $u \mapsto ux(u)$ . This product can be thought of as a CDF itself: it is weakly increasing, starts at zero, and reaches one at  $u = 1$ . Intuitively, it captures the joint effect of the type  $u$  and the probability of winning at that type.

Second, take the (right-continuous) inverse of this product map, denoted  $\psi$ .<sup>11</sup> The function  $\psi$  is continuous, weakly increasing, and always satisfies  $\psi(1) = 1$ . Its value at zero,  $\psi(0)$ , simply records the lowest type at which the original CDF turns strictly positive.

Figure 1 illustrates this transformation for a simple example. Panel (a) shows a step-shaped CDF  $x$ , panel (b) its product with  $u$ , and panel (c) the resulting  $\psi$ -transform. The picture highlights how  $\psi$  “inverts” the joint map  $ux$ . In fact, this transformation is reversible, i.e.,  $x = (\psi)^{-1}(u)/u$ , and so  $\psi$  can be thought of as a reparametrization of CDFs on  $[0, 1]$ .<sup>12</sup>

<sup>10</sup>Recall that a point in a convex set is extreme if it cannot be expressed as a convex combination of two distinct points in the set. As explained in Section 2 of Kleiner et al. [2021], by the Krein–Milman theorem any convex and compact set in a locally convex space coincides with the closed convex hull of its extreme points.

<sup>11</sup>For a CDF  $x \in \mathcal{X}$ , its right-continuous generalized inverse  $x^{-1}$  is given by  $x^{-1}(\iota) = \sup\{u \in [0, 1] : x(u) \leq \iota\}$ .

<sup>12</sup>More formally, let  $\Psi$  be the subset of  $\mathcal{X}$  that consists of continuous functions  $\psi$  such that  $\iota/\psi(\iota) \in [0, 1]$  is weakly increasing. Then,  $x \mapsto \psi$  is a bijection between  $\mathcal{X}$  and  $\Psi$ .

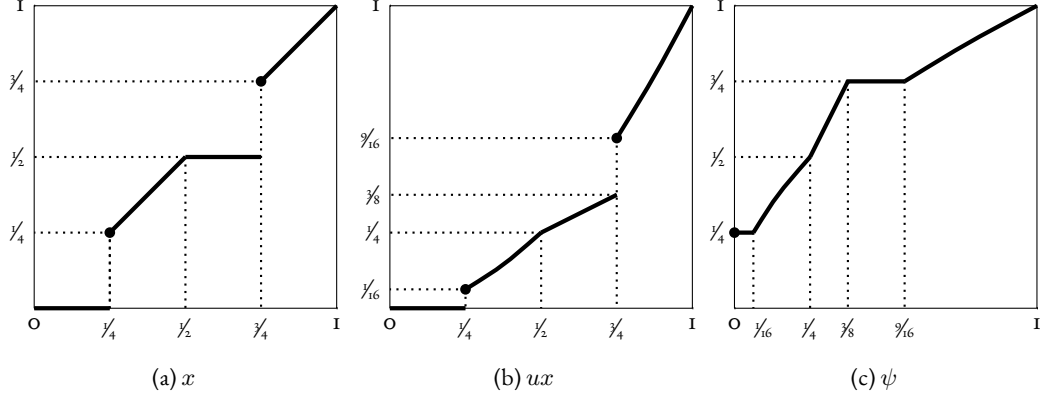


Figure 1: Illustration of the transformation  $\psi$  for  $x(u) = u \cdot \mathbf{1}_{[1/4, 1/2) \cup [3/4, 1)}(u) + 1/2 \cdot \mathbf{1}_{[1/2, 3/4)}(u)$ .

## 5.2 Feasibility through the principal curve

The  $\psi$ -transform is more than a convenient relabeling: it is the tool that allows us to collapse Border's  $n$ -dimensional condition into a unidimensional one. The principal curve  $\nu$  mentioned above is built directly from the family of transforms  $(\psi_1, \dots, \psi_n)$ , one for each bidder. Given a reduced form  $\mathbf{x}$ , which is not necessarily feasible, we begin by combining the individual transforms through their geometric mean,

$$\bar{\psi} = \sqrt[n]{\prod_{i=1}^n \psi_i},$$

and then set for each  $s \in [0, 1]$ ,

$$\nu_i(s) = \psi_i(\bar{\psi}^{-1}(s)). \quad (8)$$

The principal curve  $\boldsymbol{\nu} = (\nu_1, \dots, \nu_n)$  is continuous and weakly increasing. It begins at the minimal quantile types where bidders' interim winning probabilities are nonzero, which is precisely  $(\psi_1(0), \dots, \psi_n(0))$ , and ends at  $(1, \dots, 1)$ .

The main result of this section is the following characterization of feasibility through the principal curve explicitly defined in (8).

**Theorem 1.** *A reduced form  $\mathbf{x}$  is feasible if and only if*

$$B(\boldsymbol{\nu}(s)) \leq 1 \quad \forall s \in [0, 1].$$

Moreover, along the principal curve  $\boldsymbol{\nu}$ , the value of  $B$  can be expressed as the following function of  $\bar{\psi}$  alone:

$$B(\boldsymbol{\nu}(s)) = \left( \max \{ \bar{\psi}(0), s \} \right)^n + n \int_{\bar{\psi}^{-1}(s)}^1 \iota \, d \ln \bar{\psi}(\iota). \quad (9)$$

The logic behind the theorem is as follows. We show in the appendix that for a given  $s$ , the vector  $\boldsymbol{\nu}(s)$  maximizes the left-hand side of Border's inequality among all cutoff vectors with  $\prod_i u_i \geq s^n$ .<sup>13</sup> Thus, if the inequality holds at this "hardest" point, it automatically holds for every other cutoff vector with the same product. Checking feasibility along the principal curve therefore suffices.

<sup>13</sup>If either  $s = 0$  or  $s < \bar{\psi}(0)$ , then any  $\mathbf{u}$  such that  $\prod_{i=1}^n u_i \geq s^n$  and  $u_i \leq \psi_i(0)$  for  $i = 1, \dots, n$  is optimal.

The principal curve parametrizes the cutoffs that put the greatest strain on feasibility, and it does so in a way that depends only on the  $\psi$ -transforms of the reduced form.<sup>14</sup> Perhaps surprisingly, in our continuous-type setting with monotonicity constraints, the value of  $B$  along that curve depends on these transforms only through their geometric average  $\bar{\psi}$ , as in (9). The explicit formula for  $B(\nu(s))$  makes this reduction particularly tractable.

**Example 1.** To illustrate, consider reduced forms of the power type  $x_i(u) = u^{1/\alpha_i - 1}$ , where  $\alpha_i \in (0, 1)$ . Direct computations reveal that  $\psi_i(\iota) = \iota^{\alpha_i}$  and so  $\bar{\psi}(\iota) = \iota^{\frac{\sum_{i=1}^n \alpha_i}{n}}$ , which gives

$$B(\nu(s)) = s^n + \left(1 - s^{n/\sum_{i=1}^n \alpha_i}\right) \sum_{i=1}^n \alpha_i. \quad (10)$$

Applying the theorem, we conclude

$$x \text{ is feasible} \iff \sum_{i=1}^n \alpha_i \leq 1.$$

This unidimensional reduction provides a complete and tractable test for feasibility. Just as important, it prepares the ground for our next steps, namely the characterization of extremality and optimality that we will be covering in the subsequent sections.

## 6 Extremality

In this section, we study the structure of extremal reduced forms and the structure of corresponding allocations that induce them.

### 6.1 Extremal reduced forms

Whenever the revenue functions  $H_i$  are convex in  $x_i$ —as they are in many applications with endogenous valuations—the objective in (7) is convex. Maximizing a continuous convex function over a convex compact set always yields a solution at an extreme point. In such cases, the optimal mechanism can be implemented by an extremal reduced form. Even when the objective is not convex, extremal reduced forms remain essential serving as the “building blocks” of the feasible set, since any feasible reduced form can be expressed as a convex combination of them.<sup>15</sup>

This perspective was first emphasized in the influential paper [Kleiner et al. \[2021\]](#), which demonstrated the surprising tractability of extreme points of feasible symmetric reduced forms and showed how they illuminate problems in auctions, delegation, and persuasion. Here, we take the next natural step in this agenda by studying extreme points of (potentially asymmetric) feasible reduced forms. We show that this broader set also admits a tractable description, one that can be directly exploited in economic applications.

At an intuitive level, extremal reduced forms correspond to “corners” of the feasible set, where Border’s condition is maximally tight. If the inequality were slack anywhere, then the reduced form could be perturbed in different directions and written as a convex combination of other feasible points. Our main result in this section establishes that extremality arises precisely when the feasibility constraints along the principal curve bind at every point.

**Theorem 2.** A feasible reduced form  $x$  is extremal if and only if

$$B(\nu(s)) = 1 \quad \forall s \in [0, 1],$$

<sup>14</sup>A related observation also appeared in [Cai et al. \[2018\]](#), who studied feasibility of (potentially nonmonotone) reduced forms in settings with finitely many types.

<sup>15</sup>For example, see [Kleiner et al. \[2021\]](#), who explained that every feasible reduced form can be written as an integral over extremal ones due to the Choquet theorem.

or equivalently:

$$\bar{\psi}(\iota) = \max \{ \bar{\psi}(0), \sqrt[n]{\iota} \} \quad \forall \iota \in [0, 1]. \quad (\text{II})$$

Theorem 2 makes it straightforward to verify extremality. For example, recall the setting of Example 1, where  $B$  along the principal curve is given by (10). Direct calculations show that a reduced form of the power type,

$$\mathbf{x} \text{ is extremal} \iff \sum_{i=1}^n \alpha_i = 1.$$

The same conclusion follows immediately from (II), using the fact that  $\bar{\psi}(\iota) = \iota^{\frac{\sum_{i=1}^n \alpha_i}{n}}$ .

Although the characterization in Theorem 2 may at first appear abstract, it has a natural interpretation when viewed through the principal curve. For each bidder  $i$ 's type  $u_i = u$ , the principal curve specifies a point  $\nu(s)$  on the curve that corresponds to that type, that is

$$s = (\nu_i)^{-1}(u).$$

Extremality requires that the feasibility inequality be tight along this path, meaning that bidder  $i$  with type  $u$  wins against all opponents whose types lie below the point on the principal curve and loses against all opponents whose types lie above that point. Formally, this statement can be expressed as<sup>16</sup>

$$x_i(u) = \Pr(u_j < \nu_j(s) \quad \forall j \neq i) = \prod_{j \neq i} \nu_j(s), \quad (12)$$

Figure 2 illustrates this point with a simple numerical example. In this figure, the black line corresponds to the principal curve with  $u_1$  and  $u_2$  on the horizontal and vertical axes respectively, constructed from an extremal reduced form.

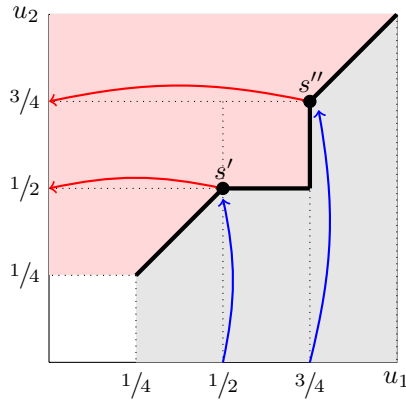


Figure 2: Principal curve (black) induced by an extremal reduced form. The points  $s', s''$  correspond to bidder 1 types  $u = 1/2, 3/4$  and the associated bidder 2's cutoffs  $\nu_2(s') = 1/2, \nu_2(s'') = 3/4$ .

For example, bidder 1 with type  $u = 1/2$  wins with probability  $1/2$  and with type  $u = 3/4$  wins with probability

<sup>16</sup>To see it, recollect that  $u x_i(u) = \psi_i^{-1}(u)$  due to the definition of the  $\psi$ -transform, and  $\prod_{j \neq i} \nu_j(s) = s^n / u = \psi_i^{-1}(u) / u$  due to the definition of the principal curve and Equation (II).

$3/4$ . Using (12), we can recover the reduced form consistent with the principal curve shown in the figure:

$$x_1(u) = u \cdot \mathbf{1}_{[1/4, 1/2) \cup [3/4, 1]}(u) + 1/2 \cdot \mathbf{1}_{[1/2, 3/4)}(u), \quad (13)$$

$$x_2(u) = \max\{1/4, u\} \cdot \mathbf{1}_{[0, 1/2)}(u) + \min\{u + 1/4, 1\} \cdot \mathbf{1}_{[1/2, 1)}(u). \quad (14)$$

This representation makes clear that the allocation inducing this extremal reduced form is unique: bidder 1 wins in the gray region, bidder 2 in the red region, and otherwise the good remains unallocated. Finally, note that the reduced form defined in (13), (14) satisfy two simple inverse relations:  $x_1 = x_2^{-1}$  on  $x_1 > 0$  and  $x_2 = x_1^{-1}$  on  $x_2 > 0$ . For the two-bidder case, Yang and Yang [2025] established that this characterization is equivalent to extremality. With more than two bidders, no such unidimensional inverse relationship exists: the coordinates of  $\mathbf{x}$  are genuinely nonlinear. Equation (12) can therefore be viewed as the natural multidimensional generalization of the two-bidder case.

## 6.2 Allocations inducing extremal reduced forms

The analysis in the previous section showed that extremal reduced forms are entirely determined by their principal curves and that Border’s condition must bind along these curves. This already indicates that such reduced forms can arise only from highly structured allocation rules. We now turn to these allocation rules. Understanding them clarifies the geometry behind extremality and will allow us to revisit—and sharpen—the classical BIC–DIC equivalence.

The BIC–DIC equivalence, first established by Manelli and Vincent [2010], states that every extremal reduced form can be implemented in dominant strategies. In other words, one can find an allocation  $\mathbf{z}$  that induces the reduced form in question such that each  $z_i$  is weakly increasing in bidder  $i$ ’s own type.<sup>17</sup> Their original proof considers piecewise-constant reduced forms together with an approximation argument. Our characterization of extremality provides a more direct route to obtain the BIC–DIC equivalence without relying on limiting arguments. We explicitly characterize how to induce extremal reduced form and establish that the allocations inducing them satisfy the aforementioned DIC property as well as two other properties that could not be seen with the earlier approach. Since explained in Footnote 15, any feasible reduced form can be expressed as an integral over extremal reduced forms, the equivalence emerges almost automatically.

To make this structure precise, we introduce the notion of a score allocation.

**Definition 1.** An allocation  $\mathbf{z}$  is a **score allocation** if there exists a tuple of strictly increasing, right-continuous functions  $\mathbf{q} = (q_1, \dots, q_n) : [0, 1] \rightarrow \mathbf{R}^n$  such that

$$z_i(\mathbf{u}) = \begin{cases} \mathbf{1}_{q_i^{-1}(\mathbf{R}_+)}(u_i) & \text{if } q_i(u_i) > \max_{j \neq i} q_j(u_j), \\ 0 & \text{otherwise.} \end{cases} \quad (15)$$

Here,  $q_i(u_i)$  is the **score** assigned to bidder  $i$  with type  $u_i$ , and the good is awarded to the bidder with the highest nonnegative score.<sup>18</sup> Score allocations provide a remarkably simple description of the complex feasible set: they are exactly the allocations that induce extremal reduced forms. Such allocations are also familiar in classical settings. For example, Myerson’s optimal auction is a score allocation whenever bidders’ MVVs are strictly increasing in their types: the mechanism that awards the object to the bidder with the highest nonnegative MVV is exactly of this

<sup>17</sup>See also Gershkov et al. [2013] and Goeree and Kushnir [2023].

<sup>18</sup>Uniqueness here refers to the allocation rule  $\mathbf{z}$ , not to the score functions  $\mathbf{q}$  themselves.

form.<sup>19</sup>

Score allocations induce extremal reduced forms. To see it, take the set of scores  $\mathbf{q}$  and remark that the score allocation  $\mathbf{z}$  is a unique (pointwise) maximizer

$$\sum_{i=1}^n \int_{[0,1]^n} q_i(u_i) \tilde{z}_i(\mathbf{u}) d\mathbf{u}$$

in the space of all allocations, where uniqueness is the direct consequence of strict monotonicity of scores. As a result, the reduced form  $\mathbf{x}$  induced by that score allocation, i.e.,  $x_i(u_i) = \mathbf{E}[z_i|u_i]$ , is a unique maximizer of

$$\sum_{i=1}^n \int_{[0,1]^n} q_i(u_i) \tilde{x}_i(\mathbf{u}) d\mathbf{u}$$

in the space of all feasible reduced forms. This shows that  $\mathbf{x}$  is not merely extremal but even exposed.<sup>20</sup> Strikingly, the converse is also true as shown in the theorem below.

**Theorem 3.** *A feasible reduced form is extremal if and only if it can be induced by a score allocation. Moreover, whenever this holds, the score allocation with scores  $q_i(u) = u x_i(u)$  on  $[\psi_i(0), 1]$  and  $q_i(u) < 0$  on its complement induces it.*

This theorem makes the link between extremality and score allocations completely transparent. First, the entire set of extremal reduced forms can be generated simply by working with scores and then computing bidders' winning probabilities. Second, to test whether a given reduced form is extremal, it suffices to construct the corresponding score function—just the type multiplied by the interim winning probability, truncated below  $\psi_i(0)$  and verify that  $\mathbf{z}$  defined in (15) integrates to the reduced form  $\mathbf{x}$  that we started with. Both tasks are strikingly simple compared to the original  $n$ -dimensional Border's condition.

Although score allocations have appeared in the social choice literature, their connection to extremality has not, to the best of our knowledge, been recognized. Mishra and Quadir [2014] referred to them as *simple utility maximizers* and showed that they can be axiomatized by three basic properties:

1. *Deterministic.* For every type profile exactly one bidder receives the good, or none if all scores are strictly negative.
2. *Monotone.* Each bidder's allocation is weakly increasing in their own type.
3. *Nonbossy.* A bidder cannot change the allocation of others without also changing their own. Formally,

$$z_i(u_i, \mathbf{u}_{-i}) = z_i(u'_i, \mathbf{u}_{-i}) \implies \mathbf{z}(u_i, \mathbf{u}_{-i}) = \mathbf{z}(u'_i, \mathbf{u}_{-i}) \quad \forall i, \mathbf{u}_{-i}.$$

In words, if bidder  $i$ 's own outcome does not change when their type is varied, then the outcome for all other bidders must also remain unchanged. This condition rules out “bossy” manipulations where the bidder could leave their own allocation fixed while reshuffling who among their rivals receives the good.

Taken together, these properties have a sharp geometric implication: the regions of the type space where the auctioneer keeps the good,  $\{\mathbf{u} \in [0, 1]^n : \mathbf{z}(\mathbf{u}) = 0\}$ , and the regions where each bidder  $i$ 's type  $u_i \in [0, 1]$

<sup>19</sup>As explained in Section 2.1, bidder  $i$ 's Myerson's virtual value as a function of their type  $\theta_i$  equals  $\theta_i - \frac{1-F_i(\theta_i)}{f_i(\theta_i)}$ . In the quantile space, this object can be expressed as  $v_i(u) - (1-u)v'_i(u)$ , where  $v_i$  is the inverse of  $F_i$ .

<sup>20</sup>Recall that a point in a convex subset of a topological vector space is exposed if some continuous linear functional attains its strict maximum at that point, and every exposed point is extremal, e.g., see Kleiner et al. [2021]. Here, the objective is a continuous linear functional when the set of CDFs on the unit interval is viewed as a subset of Lebesgue integrable functions endowed with the norm topology.

wins, the sets  $\{\mathbf{u}_{-i} \in [0, 1]^{n-1} : z_i(u_i, \mathbf{u}_{-i}) = 1\}$  are hyperrectangles aligned with the axes. This rectangular structure is exactly what makes score allocations induce extremal reduced forms. Note that a vast majority of ex post allocations do not admit this rectangular structure, e.g., suppose the auctioneer awards the good to bidder 1 when  $u_1 > \frac{u_2 + u_3}{2}$ , and otherwise allocates it efficiently among bidders 2 and 3. Clearly, the set of  $(u_2, u_3)$  such that bidder 1's type  $u_1$  wins is not a rectangle.

## 7 Optimality

Having characterized feasibility and extremality, we now turn to optimality. A central difficulty is that the auctioneer's objective is generally nonlinear in interim winning probabilities, which rules out the convenient pointwise maximization available in Myerson's linear case. Instead, we reformulate the problem as an optimal control problem that constructs the principal curve continuously. This reformulation reveals the key feature of the optimal auction—an *equalization of marginal revenues along the optimal principal curve*—and allows us to characterize the optimum in regular settings in Sections 8 and 9.

### 7.1 $\delta$ -transform

Recall that feasibility can be expressed in terms of the geometric mean of the  $\psi$ -transforms  $(\psi_1, \dots, \psi_n)$  along the principal curve. This observation naturally suggests searching for the optimum directly over these transformations that satisfy the feasibility criterion in Theorem 1. However, two challenges immediately arise. First, the objective is still expressed in terms of the original interim winning probabilities  $(x_1, \dots, x_n)$ , so both revenue and feasibility must first be written in the same language. Second, working directly with  $\psi$ -transforms is difficult because each  $\psi_i$  must be continuous, weakly increasing, and satisfy the additional structural condition that  $\iota/\psi_i(\iota) \in [0, 1]$  is also weakly increasing (see Footnote 12).

To resolve both issues simultaneously, we introduce a logarithmic reparameterization of the  $\psi$ -transform. For each  $x \in \mathcal{X}$  and  $t \geq 0$ , let

$$\delta(t) = -\ln \psi(e^{-t}),$$

which we term a  $\delta$ -transform of  $x$  for short, since  $\psi$  is derived from  $x$ . The properties of the  $\psi$ -transform established in Section 5—especially Footnote 12—imply that  $\delta$  is absolutely continuous with derivative lying between 0 and 1. Thus,  $\delta'(t)$  is well defined almost everywhere and can be interpreted as the elasticity of the  $\psi$ -transform,

Just as  $\psi$  provides a reparametrization of a CDF  $x \in \mathcal{X}$ , the function  $\delta$  provides an equivalent (and more tractable) reparametrization, and therefore we may work directly with  $\delta$  while recovering the underlying CDF when needed.<sup>21</sup> To see how the  $\delta$ -transform encodes CDFs, observe that it defines a parametric curve

$$t \in \mathbf{R}_+ \mapsto \left( e^{-\delta(t)}, e^{\delta(t)-t} \right)$$

in the unit square, where the horizontal and vertical axes correspond to type  $u$  and probability  $x$ , respectively. For any fixed type  $u$ , the highest value of the second coordinate attained on this curve when the first coordinate equals  $u$  is precisely the value of the CDF at that type  $x(u)$ , that is

$$x(u) = e^{\delta(t)-t} \Big|_{t=(\delta)^{-1}(-\ln u)}. \quad (16)$$

<sup>21</sup>Formally,  $x \mapsto \delta$  is a bijection between CDFs on the unit interval and absolutely continuous functions on  $\mathbf{R}_+$  that start at zero and have derivative in  $[0, 1]$ .



Thus, the  $\delta$ -transform captures how quickly the cumulative probability encoded in  $x$  accumulates as we move along the unit interval, starting from  $u = 1$  at  $t = 0$ . In the appendix, we elaborate more on the geometry of this transform.

## 7.2 Optimal control along the principal curve

In this section, we first use the  $\delta$ -transform to place both feasibility and revenue in the same transformed space so that the auctioneer's problem no longer mixes different representations. We then show that the auctioneer's problem can be expressed as a textbook optimal control problem with absolutely continuous state variables to which standard tools from convex analysis and continuous-time control theory apply directly.

Given a reduced form  $\mathbf{x}$  (not necessarily feasible), we claim that the auctioneer's revenue can be written as

$$\sum_{i=1}^n \int_0^\infty e^{-t} R_i(\delta_i(t), t) dt, \quad (17)$$

where  $R_i(\delta, t)$  is given by<sup>22</sup>

$$R_i(\delta, t) = \int_0^\delta \frac{\partial}{\partial x} H_i(e^{\tau-t}, e^{-\tau}) d\tau. \quad (18)$$

Economically,  $R_i(\delta, t)$  aggregates marginal contributions  $\frac{\partial}{\partial x} H_i$  of all types  $u = e^{-\tau}$  with  $\tau \in [0, \delta]$ , where each such type is evaluated at allocation probability  $x = e^{\tau-t}$  along the curve indexed by  $t$ . The outer integral then sums these cumulative marginal values across all such  $t$ -curves. To see the logic behind (17), note that for each bidder  $i$ , we have

$$\begin{aligned} \int_0^1 H_i(x_i(u), u) du &= \int_0^1 \int_0^1 \mathbf{1}_{[0, u x_i(u)]}(\iota) \frac{\partial}{\partial x} H_i\left(\frac{\iota}{u}, u\right) d\iota d \ln u \\ &= \int_0^1 \int_0^1 \mathbf{1}_{[\psi_i(\iota), 1]}(u) \frac{\partial}{\partial x} H_i\left(\frac{\iota}{u}, u\right) d \ln u d\iota = \int_0^\infty e^{-t} R_i(\delta_i(t), t) dt, \end{aligned}$$

where we used the definition of the  $\psi$ - and  $\delta$ -transforms, and (18).

As for feasibility, recollect that  $\mathbf{x}$  is feasible if and only if Border's constraints hold along the principal curve. Using (9) from Theorem 1, we can express feasibility of  $\mathbf{x}$  through its  $\delta$ -transforms (see the appendix for details) as

$$\int_0^t e^{-\tau} \sum_{i=1}^n \delta'_i(\tau) d\tau \leq 1 - e^{-\sum_{i=1}^n \delta_i(t)}. \quad (19)$$

Economically, (19) plays the role of a dynamic feasibility constraint. The left-hand side records how much the total winning probability has been allocated up to the point where the principal curve reaches types at quantile  $e^{-t}$ . The right-hand side records how much probability could possibly have been allocated by that point, given the cumulative height  $\sum_{i=1}^n \delta_i(t)$  of the principal curve.

Putting everything together—the expression for revenue in terms of  $\delta$  and the feasibility condition in the same variables—we obtain the following representation of the auctioneer's problem.

<sup>22</sup>Since  $H_i(x, u) = J_i(x, F_i^{-1}(u))$ , where  $J$  is twice continuously differentiable and  $F_i$  is continuously differentiable with  $f_i > 0$ , the function  $R_i$  is well-defined. In fact, it is continuously differentiable with  $R_i(0, t) = 0$  and uniformly bounded  $\frac{\partial}{\partial \delta} R_i$ .

**Proposition 1.** *A reduced form is optimal if and only if its  $\delta$ -transforms solve the following program:*

$$\max_{\delta} \int_0^\infty e^{-t} \sum_{i=1}^n R_i(\delta_i(t), t) dt \quad \text{s.t.} \quad \delta' \in [0, 1]^n, \quad \delta(0) = \mathbf{0}, \quad \text{and (19)}. \quad (20)$$

*Moreover, there exists an optimal reduced form that is extremal if and only if there exists a solution to (20) satisfying*

$$\sum_{i=1}^n \delta'_i(t) = \mathbf{1}_{[0, T]}(t) \quad \text{for some } T \in [0, \infty].$$

The first part of the proposition formalizes how the abstract maximization problem (7) can be rewritten in a transparent way as an optimal control problem along the principal curve. The second part highlights a sharp structural implication when searching for the optimum among extremal reduced forms.<sup>23</sup> Our characterization of extremality in Theorem 2 implies that any extremal reduced form corresponds to a front-loaded allocation path in continuous time in which probability is assigned at the maximal feasible rate up to a cutoff, after which allocation ceases. Hence, when solving (20), it suffices to search among policies satisfying  $\sum_{i=1}^n \delta'_i(t) = 1$  until some cutoff time  $T \in [0, \infty]$ , followed by no further allocation.

The maximization problem that appears in the proposition is an optimal control problem with pure state constraints as defined in Chapter 6 of Seierstad and Sydsæter [1986]. To see this, note that (19) can be written as

$$\mu(t) \leq 1 - e^{-\sum_{i=1}^n \delta_i(t)},$$

where  $\mu$  is an auxiliary state variable with dynamics  $\mu'(t) = e^{-t} \sum_{i=1}^n a_i(t)$ ,  $\mu(0) = 0$ , and  $\mathbf{a} = \delta' \in [0, 1]^n$  is the control vector. The existence and characterization of its solution through the Pontryagin maximum principle directly follow from results in this book. Here, we focus on a single necessary condition—derived from first principles in the appendix—that is particularly informative about the structure of the optimum.

**Proposition 2.** *Let  $\mathbf{x}^*$  be optimal and denote its  $\delta$ -transforms by  $\delta^*$ . Consider an interval  $(\underline{t}, \bar{t})$  and two distinct bidders  $i \neq j$  such that both  $(\delta_i^*)', (\delta_j^*)'$  are bounded away from 0 and 1 on that interval. Then, their marginal revenues are equalized:*

$$\frac{\partial}{\partial \delta} R_i(\delta_i^*(t), t) = \frac{\partial}{\partial \delta} R_j(\delta_j^*(t), t) \quad \forall t \in (\underline{t}, \bar{t}).$$

Recall that  $\delta^*$  defines the principal curve, and that intervals on which some  $(\delta_i^*)'(t)$  hits the bounds  $\{0, 1\}$  correspond to jumps or flat segments of the associated interim winning probability—both arising from binding monotonicity constraints and thus from ironing. Proposition 2 identifies what happens on segments where ironing is *not* required.

*Whenever the optimal principal curve moves simultaneously in coordinates  $i$  and  $j$ , the movement must follow a direction along which their marginal revenues coincide.*

If one direction provided a strictly larger marginal return, the auctioneer would reallocate an infinitesimal amount of movement toward that coordinate and away from the other. Thus, on regions in which the interim winning probabilities are strictly increasing and continuous, the principal curve traces an “indifference path” of marginal revenues. As we explain next, this condition generalizes the familiar Myersonian rule—namely, that the object is awarded to the bidder with the highest MVV on the regular region—to nonlinear environments.

The marginal revenue equalization property in Proposition 2 therefore “almost” characterizes the optimum in settings where ironing is not required. The only missing element is the trajectory of the cumulative height  $\sum_{i=1}^n \delta_i(t)$

<sup>23</sup> As explained in Section 6, this restriction is without loss of generality when each  $H_i$  is convex in  $x_i$ .

of the principal curve, which is constrained by feasibility through (19). To make further progress and determine this trajectory at the optimum, we first study environments in which the optimum is extremal, and thus it satisfies  $\sum_{i=1}^n \delta_i(t) = t$  along the principal curve, as implied by Proposition 1. This class includes both Myerson's linear model and many auction environments with endogenous valuations. We then extend our analysis to environments in which the optimum is non-extremal as happens when bidders' have CRA preferences with positive risk-aversion.

## 8 Optimal auctions: the case of extremality

Before turning to nonlinear settings, it is instructive to revisit the classical Myerson's model through the lens of our transformed optimal-control representation.

### 8.1 Linear utilities

Recall that bidders' types in the quantile space  $u$  are related to their actual types  $\theta$  via  $\theta_i = v_i(u_i)$ , where  $v_i = F_i^{-1}$ . Thus, their revenue functions can be written as

$$H_i(x, u) = \left( \underbrace{v_i(u) - (1-u)v_i'(u)}_{=\zeta_i(u)} \right) x. \quad (21)$$

In this expression,  $\zeta_i$  is bidder  $i$ 's MVV expressed in the quantile space. The transformation in (18) reparametrizes this as

$$R_i(\delta, t) = \int_0^\delta \zeta_i(e^{-\tau}) d\tau, \quad (22)$$

and  $R_i$  is independent of  $t$  precisely because  $H_i$  is linear in its first argument.

We now use Propositions 1 and 2 to explain how our approach can be used to find the optimal auction when MVVs are strictly increasing. In this regular environment, it is natural to conjecture that the optimal principal curve moves strictly in all coordinates. If so, then Proposition 2 implies that the marginal revenues of all bidders must coincide on the entire principal curve. In addition, the second part of Proposition 1 implies that feasibility binds pointwise along the principal curve, that is  $\sum_{i=1}^n \delta_i(t) = t$ . These two observations lead us to study the following system of  $n + 1$  equations in  $n + 1$  variables:

$$\frac{\partial}{\partial \delta} R_i(\delta_i, t) = p, \quad \sum_{i=1}^n \delta_i = t, \quad (23)$$

where  $p$  can be interpreted as the common marginal revenue. It is easy to see that this system admits a unique solution, and its solution turns out to characterize Myerson's optimal auction exactly.

**Proposition 3.** *In Myerson's linear model, assume bidders' MVVs are strictly increasing. For every  $t \geq 0$ , there exists a unique solution  $(\delta^\sharp(t), p^\sharp(t))$  to the system (23), and define the cutoff time*

$$T = \inf\{t \geq 0 : p^\sharp(t) \leq 0\}.$$

*Then, the auctioneer's problem admits an optimal reduced form  $x^*$  whose  $\delta$ -transforms follow this candidate path until*

the cutoff and then remain constant.<sup>24</sup>

$$\delta_i^*(t) = \begin{cases} \delta_i^\#(t) & t \leq T, \\ \delta_i^\#(T) & \text{otherwise.} \end{cases} \quad (24)$$

By Proposition 1, each optimal reduced form corresponds to a solution to the optimal control problem in that proposition. Building on this observation, Proposition 3 identifies the candidate optimum by solving, at each  $t$ , the equalization of marginal revenues together with the aggregate constraint  $\sum_{i=1}^n \delta_i = t$ . Once these functions  $(\delta^\#, p^\#)$  are obtained, the solution to the control problem is constructed by “freezing”  $\delta^\#$  at the first time  $T$  when the common marginal revenue becomes negative. To verify that this candidate is indeed optimal, we use a dual certificate explicitly constructing dual variables to the constraint that appear in the optimal control problem.

The optimal reduced form  $\mathbf{x}^*$  is then recovered from  $\delta^*$  identified in this proposition by inverting  $(\delta_1^*, \dots, \delta_n^*)$  as in (16). To see the implications of this construction in the linear model, note that the mapping

$$t \in \mathbf{R}_+ \mapsto (e^{-\delta_1^*(t)}, \dots, e^{-\delta_n^*(t)})$$

defines a continuous, strictly decreasing curve in the  $\mathbf{u}$ -space, which starts at  $(1, \dots, 1)$  and goes all the way to  $(e^{-\delta_1^*(T)}, \dots, e^{-\delta_n^*(T)})$ . This curve coincides with the principal curve associated with  $\mathbf{x}^*$ , up to a reparametrization of its time index.

To recover the optimal principal curve  $\nu^*$  in the original parametrization, we use (23) and the fact bidder  $i$ ’s marginal revenue coincides with that bidder’s MVV, i.e.,  $\frac{\partial}{\partial \delta} R_i = \zeta_i(e^{-\delta})$ , as appears in (22). It follows that the optimal principal curve equalizes bidders’ MVVs pointwise,

$$\zeta_1(\nu_1^*(s)) = \dots = \zeta_n(\nu_n^*(s)) \quad \forall s \in [0, 1].$$

Combining this observation with (12) implies that the optimal allocation can be implemented as a score allocation. Specifically, bidder  $i$ ’s score is given by

$$q_i^*(u_i) = p^\# \left( (\delta_i^\#)^{-1}(-\ln u_i) \right) \quad (25)$$

$$= \zeta_i(u_i), \quad (26)$$

where the equality in the second line is due to (22) and (23). This confirms that our construction reproduces exactly the allocation rule originally identified in Myerson [1981].

## 8.2 Main result

We now apply our tools to nonlinear environments. A key observation is that, under an appropriate regularity condition, the optimal reduced form can still be constructed from the same auxiliary static system (23) that characterizes the Myersonian optimum in the linear case. Formally, we say that an environment is **regular** if

- (A) each  $R_i$  is strictly concave in its first argument;
- (B) for each time  $t \geq 0$ , the solution  $(\delta^\#(t), p^\#(t))$  to (23) is unique and is such that the path  $\delta^\#$  is strictly increasing and the common marginal revenue  $p^\#$  is strictly decreasing.

<sup>24</sup>In fact, this is the only optimal reduced form because the optimal control problem in (20) admits at most one solution due to the strict concavity of each  $R_i$  in its first argument.

In Myerson's linear model, Condition (A) means that bidders' virtual valuations are strictly increasing. In this case, we can invert  $\frac{\partial}{\partial \delta} R_i(\delta_i, t) = p$  directly and solve for  $\delta_i$  as a function of time  $t$  and the common marginal revenue  $p$ , which is then pinned down by  $\sum_{i=1}^n \delta_i = t$ . Condition (B) holds trivially in the linear case because each  $R_i$  is independent of  $t$ . In general nonlinear environments, Condition (B) does not follow from Condition (A). Thus, our definition of regularity is a genuine extension of Myersonian regularity to nonlinear settings. The appendix provides primitive sufficient conditions guaranteeing (A)–(B).

Then, we obtain the following characterization of the optimum.

**Theorem 4.** *Consider a regular environment, and define a cutoff time  $T$  as in Proposition 3. If for each bidder  $i$ , the function  $\delta_i^\dagger : [T, \infty) \rightarrow \mathbf{R}$  defined by  $\frac{\partial}{\partial \delta} R_i(\delta_i^\dagger(t), t) = 0$  is weakly decreasing, then the auctioneer's problem admits an optimal reduced form  $\mathbf{x}^*$ , which is extremal, and whose  $\delta$ -transforms  $\boldsymbol{\delta}^*$  take the form in Equation (24). Furthermore, this reduced form can be induced by the score allocation with scores  $\mathbf{q}^*$  of the form in Equation (25).*

Theorem 4 explicitly constructs the optimal auction in regular nonlinear environments under an additional monotonicity condition that ensures extremality of the optimum. This added condition guarantees that once bidder  $i$ 's marginal revenue falls to zero, it never becomes profitable to resume allocating probability to that bidder at lower quantiles. It is straightforward to verify that this condition holds whenever  $H_i$  is convex in its first argument; consequently, the theorem applies to a broad class of environments with endogenous valuations. It is strictly more general than convexity of  $\frac{\partial}{\partial x} H_i$  and it holds vacuously whenever one bidder's marginal revenue is nonnegative, i.e.,  $T = \infty$ , so that there no exclusion is optimal. Strikingly, despite the presence of nonlinearities that render standard approaches inapplicable, the resulting characterization of the optimum coincides formally with that obtained in the linear model.

From an economic perspective, the theorem delivers a closed-form expression for the optimal scores  $\mathbf{q}^*$ . Bidder  $i$ 's score  $q_i^*$  is given by the common marginal revenue  $p^\#(t)$  evaluated at the time corresponding to that bidder's type, namely  $t = (\delta_i^\#)^{-1}(-\ln u_i)$ . We refer to these scores as **principal virtual values (PVVs)** henceforth to distinguish them from MVVs.

The allocation based on PVVs awards the good to the bidder who lies higher on the optimal principal curve. In linear environments, PVVs coincide with MVVs, as shown in the previous section. In nonlinear environments, however, bidder  $i$ 's PVV—unlike MVV—incorporates information about other bidders. This reflects the fact that PVVs are constructed to equalize marginal revenues across tied bidders. To see it, remark that whenever the optimal interim allocation  $x_i^*(u_i)$  is strictly positive, bidder  $i$ 's PVV is simply the marginal revenue of that bidder in the optimal mechanism, that is <sup>25</sup>

$$q_i^*(u_i) = \frac{\partial}{\partial x} H_i(x_i^*(u_i), u_i).$$

In the context of the endogenous-valuation model, the marginal revenue  $\frac{\partial}{\partial x} H_i$  internalizes the effect of interim winning probabilities on bidders' investment incentives. Equalizing their marginal revenues therefore ensures that winning probability is allocated so as to balance allocative efficiency against the distortionary effects of excessive duplication of investment costs. In this sense, the optimal mechanism disciplines ex ante investment by reallocating probability until all active bidders contribute equally at the margin.

We now revisit our motivating example through the lens of Theorem 4, and then further elaborate on the economic content of the optimal mechanism when bidders' valuations are endogenous in the section that follows.

**Example 2.** *We now use Theorem 4 to verify the optimality of the score allocation described in the example in Section 3. In the parametric specification of this example, bidder  $i$ 's revenue function in the quantile space is given by  $H_i(x, u) =$*

<sup>25</sup>This representation can be obtained by combining the definition of  $R_i$  with (23), and noting that  $x_i^*(u_i)u_i = e^{-(\delta_i^\#)^{-1}(-\ln u_i)}$  (due to (16)) along the principal curve.

$u^{1/\beta_i} x^2$ , which yields

$$\frac{\partial}{\partial \delta} R_i(\delta, t) = 2e^{-(1/\beta_i - 1)\delta} e^{-t}.$$

Clearly,  $R_i$  is strictly concave provided that  $\beta_i \in (0, 1)$ .

Direct calculations show that (23) admits a unique solution of the form:

$$\delta_i^\#(t) = \kappa(1/\beta_i - 1)^{-1}t, \quad p^\#(t) = 2e^{-(1+\kappa)t},$$

where  $\kappa$  is the normalization constant so that  $\sum_{i=1}^n \delta_i^\#(t) = t$ . This shows that the environment is regular and that the cutoff time defined in Theorem 4 is infinite. As a result, the additional monotonicity condition imposed in the theorem holds trivially, hence  $\delta^* = \delta^\#$ .

Using (25), we can now recover the corresponding PVVs:

$$q_i^*(u_i) = 2u_i^{(1+1/\kappa)(1/\beta_i - 1)}.$$

Since the exponent in the first bracket is strictly positive, bidder  $i$ 's PVV exceeds  $j$ 's if and only if  $u_i^{1/\beta_i - 1} > u_j^{1/\beta_j - 1}$ . Returning to the original type space, this shows that the optimal allocation awards the good to the bidder with the highest value of  $(F_i(\theta_i))^{1/\beta_i - 1}$ , as claimed.

### 8.3 Application to discrimination in the endogenous-valuation model

In Example 2, we have argued that the optimal mechanism may be “more asymmetric” than what the auction based on MVVs may suggest. We now show using Theorem 4 that this phenomenon holds beyond that single example and is driven by the strict convexity of bidders' revenue functions in their interim winning probability.

Consider a setting with endogenous valuations where the reduced-form utility of every bidder  $i$  is equal to  $\omega_i(x_i, \theta_i) = \theta_i h(x_i)$  where  $h', h'' > 0$ . As explained in Section 2.3, bidder  $i$ 's revenue function  $J_i$  takes the form:

$$J_i(x_i, \theta_i) = \left( \theta_i - \frac{1 - F_i(\theta_i)}{f_i(\theta_i)} \right) h(x_i),$$

where the term in brackets is bidder  $i$ 's MVV, which is assumed to be strictly increasing to avoid dealing with ironing that does not add much economic insight. This setting generalizes that of Example 2 and is chosen because Myerson's virtual values arise naturally as scaling factors of MVVs even though the utilities are nonlinear.<sup>26</sup> As a result, we can naturally compare bidders' strength within this framework in terms of their MVVs and contrast the optimal auction here to Myerson's auction that allocates the good to the bidder with the nonnegative highest MVV.

One natural order that has been employed to rank bidders' strength (e.g., see Carroll and Segal [2019]) is the *hazard ratio order*. A cumulative distribution function  $F$  dominates another cumulative distribution function  $G$  in this order denoted  $F \geq_{hr} G$  when their cumulative survival functions satisfy

$$(1 - F(\theta'))(1 - G(\theta)) \geq (1 - F(\theta))(1 - G(\theta')) \quad \forall \theta \leq \theta',$$

see Chapter 1 in Shaked and Shanthikumar [2007]. As explained in this book, this ranking implies  $\frac{1-F}{f} \geq \frac{1-G}{g}$  over the common support of these two distributions. Intuitively, high types are more likely under  $F$  than under  $G$  conditional on any interval of the form  $[\theta, \infty)$ , and so a bidder whose types are drawn from  $F$  should receive a higher information rent than the one whose types are drawn from  $G$ . As a result, if there are two distinct bidders

<sup>26</sup>The primitive utility over actions-type  $(a_i, \theta_i)$  pairs that gives rise to such an  $\omega_i$  can be recovered via  $\min_x (\theta_i h(x) + a_i)/x$  when the investment costs are normalized to  $C_i(a_i) = a_i$ , see Andreyanov et al. [2026].

$i, j$  with the same value  $\theta_i = \theta_j$  and  $F_i \geq_{hr} F_j$ , then Myerson's auction will allocate the good more often to the weaker bidder  $j$ .

As shown in the previous section, to determine the optimum, one needs to work in the quantile space of bidders' types. The hazard ratio order however is silent about bidders' strength in the quantile space, where their MVVs can be expressed using  $v_i = F_i^{-1}$  analogously to (21) as

$$\zeta_i(u) = v_i(u) \left( 1 - (1-u) \frac{v'_i(u)}{v_i(u)} \right). \quad (27)$$

This happens because even if there are two distinct bidders  $i, j$  with the same quantile  $u_i = u_j$  and  $F_i \geq_{hr} F_j$  so that  $v_i \geq v_j$ , the second term in (27) can be lower for bidder  $i$  than bidder  $j$  when  $F_i$  is much more unequal than  $F_j$ . In our comparison of the optimal and Myerson's auctions, we use one more standard stochastic ordering allowing to rank bidders' strengths in the quantile space as well.

Following Chapter 4 in Shaked and Shanthikumar [2007], we say that a cumulative distribution function  $F$  dominates another cumulative distribution function  $G$  in the *star order* denoted  $F \geq_* G$  when their quantile functions satisfy

$$F^{-1}(u')G^{-1}(u) \geq F^{-1}(u)G^{-1}(u') \quad \forall u \leq u'.$$

The star order means  $G$  that is more equal relative to its mean than  $F$ . There are many parametric families of distributions for which  $F \geq_{hr} G$  automatically implies that  $G \geq_* F$ , e.g., this is true for (1) Normal, Logistic, Gumbel distributions ordered by a location parameter, (2) Exponential distributions ordered by a scale parameter, (3) Power distributions ordered by a shape parameter, as well as (4) distributions of the form in Example 2.

Going back to our auction framework, if for two bidders  $i$  and  $j$ , in addition to  $F_i \geq_{hr} F_j$ , we have  $F_j \geq_* F_i$ , then not only bidder  $i$ 's values,  $v_i$ , are stochastically higher but also per-unit information rents,  $(1-u) \frac{v'_i}{v_i}$ , are lower. This means that the stronger bidder  $i$  is allocated the good more often in the quantile space as opposed to bidder  $j$ . Equipped with these two approaches to rank bidders' strengths, we can now state our result formally.

**Proposition 4.** *Consider the model as above, and assume that the environment is regular and that bidders' MVVs are strictly increasing, and that*

$$F_1 \geq_{hr} F_i \geq_{hr} F_n, \quad F_n \geq_* F_i \geq_* F_1 \quad \forall i.$$

*Then, the optimal reduced form  $\mathbf{x}^*$  and the reduced form induced by Myerson's auction  $\mathbf{x}^M$  satisfy:*

1. *The strongest bidder 1 is always favored by the optimum relative to Myerson's auction,  $x_1^* \geq x_1^M$ ;*
2. *The weakest bidder  $n$  is always discriminated against by the optimum relative to Myerson's auction,  $x_n^* \leq x_n^M$ .*

Proposition 4 compares the interim winning probabilities of the strongest and weakest bidders across two auctions—optimal and Myerson's—and shows that the optimum allocates more often the strongest bidder at the expense of the weakest one. Intuitively, this happens in order to decrease duplication of costs that necessarily arises when bidders make investments at the ex ante stage. It is instructive to unpack the argument behind this proposition to see the economic mechanism at play as well as the role of its assumptions. Below, we sketch the argument and intuition for the first part of Proposition 4 and relegate all remaining details to the appendix.

As explained in the previous section, Myerson's auction equates bidders' MVVs along its principal curve, meaning that the  $\delta$ -transforms of  $\mathbf{x}^M$  satisfy

$$\zeta_1(e^{-\delta_1^M}) = \zeta_i(e^{-\delta_i^M}) \quad \forall i \neq 1.$$

Since the strongest bidder's distribution  $F_1$  is more equal in the sense of the star order than any other bidder  $i$ 's distribution  $F_i$ , their MVVs satisfy  $\zeta_1 \geq \zeta_i$  whenever  $\zeta_i \geq 0$ . As a result, bidder 1's  $\delta$ -transform corresponding to Myerson's auction is the highest one.

By equating MVVs, Myerson's auction completely ignores the curvature of bidders' revenue functions, which is due to the fact that interim winning probabilities only affect bidders' ex ante investment decisions, leaving bidder 1 with the highest marginal revenue along its principal curve:

$$\zeta_1 \left( e^{-\delta_1^M} \right) h' \left( e^{\delta_1^M - t} \right) \geq \zeta_i \left( e^{-\delta_i^M} \right) h' \left( e^{\delta_i^M - t} \right) \quad \forall i \neq 1. \quad (28)$$

This suggests that the auctioneer can raise expected revenue by reallocating more winning probability to bidder 1. Since the cumulative height along the principal curve is fixed at  $t$ , i.e.,  $\sum_{i=1}^n \delta_i^M(t) = t$ , the only such reallocation must be at the expenses of other weaker bidders. But this precisely what is the optimum does—it equates PVVs along its principal curve internalizing the curvature in  $h$  so that

$$\zeta_1 \left( e^{-\delta_1^*} \right) h' \left( e^{\delta_1^* - t} \right) = \zeta_i \left( e^{-\delta_i^*(t)} \right) h' \left( e^{\delta_i^*(t) - t} \right) \quad \forall i \neq 1, \quad (29)$$

and the cumulative height stays the same:  $\sum_{i=1}^n \delta_i^*(t) = t$ . Since bidders' marginal revenue is strictly concave (due to regularity), we should have  $\delta_1^* \geq \delta_1^M$ , which is equivalent to the first point in Proposition 4.

As can be seen from our argument, the only place where the star ordering is used is to conclude that bidder 1's is favored in the quantile space in Myerson's auction, which is the same as  $\delta_1^M \geq \delta_i^M$ . So, the conclusion of this proposition holds under much weaker conditions; in fact,  $x_1^*(u) \geq x_1^M(u)$  for all high enough  $u$  without any extra structure. On the other hand, not much can be said at this level of generality about the bidders other than the strongest and the weakest one because relative to Myerson's auction, the optimum simultaneously reallocates interim winning probability from them (to the strongest bidder) and to them (from the weakest bidder). Which force dominates depends on those intermediate bidders' types and fine details of the model.

## 9 Optimal auctions: going beyond extremality

Thus far, we have constructed the optimal auction in regular environments in which the optimum is extremal, so that the cumulative height satisfies  $\sum_{i=1}^n \delta_i(t) = t$  along the corresponding principal curve. As shown in Theorem 2, non-extremal reduced forms violate this property. Nevertheless, such violations can be optimal for the auctioneer even under regularity when  $H_i$  exhibits satiation with respect to their interim winning probabilities. In these environments, rather than excluding bidders from trade once marginal revenue becomes nonpositive, the auctioneer may optimally continue allocating the good while keeping  $\sum_{i=1}^n \delta_i(t) < t$  at the bottom of the principal curve.

To illustrate the economic force behind non-extremality, consider an IPV model with CRA preferences and a single bidder whose certainty equivalent is quadratic, and whose type is uniformly distributed on the unit interval, so that  $H(x, u) = xu - (1-u)x^2$ . With a single bidder, the feasibility constraint is vacuous, and the problem reduces to maximizing  $\int_0^1 H(x(u), u)$  in the space of CDFs. Due to strict concavity and supermodularity, the optimum  $x^*$  can be obtained using the pointwise first-order condition that gives

$$x^*(u) = \min \left\{ \frac{1}{2} \frac{u}{1-u}, 1 \right\}.$$

For low types, the optimal allocation assigns winning probability strictly between zero and one. This interior solution reflects the incentive value of reducing information rents by exploiting bidder's risk aversion. We refer the



reader to GMSZ-CRA for an excellent treatment of such settings in the  $x$ -space and additional economic intuition, and highlight their implications for the optimal control in the  $\delta$ -space.

Direct calculations show that  $\frac{\partial}{\partial \delta} R(\delta, t) = e^{-\delta} - 2(1 - e^{-\delta})e^{\delta-t}$  so that the auctioneer's optimal control problem in Proposition 1 can be restated as

$$\max_{\delta} \int_0^{\infty} e^{-t} R(\delta(t), t) dt \quad \text{s.t.} \quad \delta' \in [0, 1], \quad \delta(0) = 0.$$

Relaxing this problem by replacing  $\delta'(t) \in [0, 1]$  with the weaker requirement of  $\delta(t) \in [0, t]$ , one can again solve such relaxation using the pointwise first-order conditions and obtain  $\delta^*(t) = t$  with  $\frac{\partial}{\partial \delta} R(\delta^*(t), t) > 0$  below a certain threshold and  $\delta^*(t) < t$  with  $\frac{\partial}{\partial \delta} R(\delta^*(t), t) = 0$  above that threshold. This latter region corresponds precisely to types for which the induced interim winning probability lies strictly between zero and one. Thus, in contrast to the extremal case studied above—where the optimal policy freezes once marginal revenue becomes non-positive—here, the principal curve continues to move while the winning probability is smoothly reduced.

The same phenomenon can arise with multiple (possibly asymmetric) bidders. In such cases, the optimal mechanism may again be non-extremal, with probability allocated at a strictly lower rate than the feasibility frontier allows. Nevertheless, the structure uncovered in the extremal case remains informative: optimal allocations continue to be organized by marginal revenue equalization, but are implemented through more flexible allocation rules. In what follows, we formalize this idea using fractional score allocations, which generalize score-based mechanisms by allowing the auctioneer to scale winning probabilities while preserving the ranking induced by scores.

**Definition 2.** An allocation  $z$  is a **fractional score allocation** if there exists a tuple of scores  $\mathbf{q}$  in the sense of Definition 1 and a tuple of right-continuous functions  $\mathbf{r} = (r_1, \dots, r_n) : [0, 1] \rightarrow [0, 1]^n$  such that

$$z_i(\mathbf{u}) = \begin{cases} r_i(u_i) & \text{if } q_i(u_i) > \max_{j \neq i} q_j(u_j), \\ 0 & \text{otherwise.} \end{cases} \quad (30)$$

Fractional score allocations naturally generalize score allocations by allowing bidder  $i$ 's **winning fraction**  $r_i(u_i)$  to be different from  $\mathbf{1}_{q_i^{-1}(\mathbf{R}_+)}(u_i)$ . That is, the ranking of bidders is still determined by scores, but the probability with which the highest-ranked bidder receives the object may be strictly less than one. Equipped with this definition, we can now characterize the optimal mechanism beyond extremality

**Theorem 5.** Consider a regular environment, and define a cutoff time  $T$  as in Proposition 3. If for each bidder  $i$ , the function  $\delta_i^\dagger : [T, \infty) \rightarrow \mathbf{R}$  defined by  $\frac{\partial}{\partial \delta} R_i(\delta^\dagger(t), t) = 0$  is weakly increasing with the derivative of at most one, then the auctioneer's problem admits an optimal reduced form  $\mathbf{x}^*$  whose  $\delta$ -transforms  $\delta^*$  coincide with  $\delta^\#$  on  $[0, T)$  and  $\delta^\dagger$  on  $[T, \infty)$ .<sup>27</sup> Furthermore, this reduced can be induced by the fractional score allocation with scores  $\mathbf{q}^*$  of the form in Equation (25) and fractions  $\mathbf{r}^*$  given by

$$r_i^*(u_i) = \begin{cases} 1 & u_i \geq e^{-\delta_i^*(T)}, \\ \frac{x_i^\dagger(u_i)}{x_i^\#(u_i)} & \text{otherwise,} \end{cases} \quad (31)$$

where  $x_i^\#$  is the interim winning probability induced by the score allocation with scores  $\mathbf{q}^*$ , and  $x_i^\dagger$  is obtained by reverting  $\delta_i^\dagger$  using (16).<sup>28</sup>

<sup>27</sup>Since  $p^\#(T) = 0$ , we have  $\delta^\#(T) = \delta^\dagger(T)$ , and so  $\delta^*$  is absolutely continuous due to uniqueness of both  $\delta^\#$ ,  $\delta^\dagger$  under regularity.

<sup>28</sup>As a convention, we set  $0/0 = 0$  in (31).

Theorem 5 extends the characterization of optimal auctions to regular environments in which the optimal reduced form is not extremal. The additional monotonicity condition imposed on the function  $\delta_i^\dagger$  ensures that, once bidder  $i$ 's marginal revenue reaches zero, it becomes optimal to progressively reduce the rate at which winning probability is assigned to that bidder. It is easy to see that this condition holds whenever  $H_i$  is concave in its first argument and supermodular, and thus Theorem 5 applies to the large class of auction models with CRA preferences introduced in GMSZ-CRA.

Economically, the optimum takes the form of a fractional score allocation with scores constructed from (23) and fractions given by (31).<sup>29</sup> This structure implies that bidder  $i$ 's interim winning probability satisfies

$$x_i^*(u) = \begin{cases} x_i^\#(u) & u \geq e^{-\delta_i^*(T)}, \\ x_i^\dagger(u) & \text{otherwise,} \end{cases}$$

where  $x_i^\#$  is the probability induced by the extremal score allocation and  $x_i^\dagger$  is obtained from  $\delta_i^\dagger$  via (16). It follows directly from the first-order condition defining  $\delta_i^\dagger$  that  $x_i^\dagger(u)$  coincides with the pointwise maximizer of  $H_i$  for types below the threshold, exactly as in the single-bidder benchmark used to motivate the result.

The mechanism therefore operates in two distinct regimes. At high quantiles, it allocates probability as in the extremal case, equalizing marginal revenues across bidders along the principal curve. Once the common marginal revenue reaches zero, the allocation smoothly transitions to the bidder-specific optimum  $x_i^\dagger$  while preserving the ranking induced by scores. Consequently, unlike the extremal case—where allocation ceases entirely once marginal revenue becomes non-positive—low types may continue to receive the good with strictly positive probability.

We elaborate on this point in the next section, where we examine the optimal pattern of discrimination in the CRA-preferences model.

## 9.1 Application to discrimination in the CRA-preferences model

In this section, we consider the CRA-preferences model and use Theorem 5 to study how the optimal mechanism favors bidders as a function of their risk attitudes. It turns out that the optimal allocation discriminates against more risk-averse bidders at high types and against less risk-averse bidders at low types.

To make this comparison as transparent as possible, we assume that all bidders share the same type distribution  $F$  with inverse  $v = F^{-1}$ . Bidders belong to one of two groups: risk-neutral and risk-averse. A risk-neutral bidder has a linear certainty equivalent equal to  $x$ , whereas a risk-averse bidder has a strictly convex certainty equivalent  $g(x) \leq x$  satisfying  $g(0) = 0$ ,  $g(1) = 1$ ,  $g' > 0$ , and  $g'' > 0$ . There are  $k$  risk-neutral bidders, indexed by  $i = 1, \dots, k$ , and  $n - k$  risk-averse bidders, indexed by  $i = k + 1, \dots, n$ .

Because the environment is symmetric within each group, the optimal mechanism is group-symmetric as well. Hence, it suffices to characterize the interim winning probabilities (and their  $\delta$ -transforms) for a representative risk-neutral bidder, denoted  $i = 1$ , and a representative risk-averse bidder, denoted  $i = n$ . As explained above, the extremal candidate is obtained by solving (23) for  $\delta_1^\#$ ,  $\delta_n^\#$ , and the common marginal revenue  $p^\#$ :

$$p = v(e^{-\delta_1}) - v'(e^{-\delta_1})(1 - e^{-\delta_1}) = v(e^{-\delta_n}) - v'(e^{-\delta_n})(1 - e^{-\delta_n})g'(e^{\delta_n - t}), \quad (32)$$

and

$$t = k\delta_1 + (n - k)\delta_n. \quad (33)$$

<sup>29</sup>The optimal allocation is not unique unless it is extremal; implementing it through fractional score allocations is one convenient way to induce the optimal reduced form.

Once this solution is constructed, we can similarly obtain the continuation paths  $\delta_1^\dagger, \delta_n^\dagger$  and invoke Theorem 5 to recover the optimal interim winning probabilities  $x_1^*, x_n^*$ .

We are interested in comparing  $x_1^*$  and  $x_n^*$ . Since they are determined through the marginal revenue equalization condition, the optimal pattern of discrimination depends on comparing  $\frac{\partial}{\partial x} H_i$ , and thus on comparing  $x' = 1$  with  $g'$ . In principle, the difference  $1 - g'(x)$  may change sign arbitrarily many times for intermediate values of  $x$ . To gain traction and deliver a global result, we henceforth assume that  $x - g(x)$  is unimodal, attaining its maximum at some  $m \in (0, 1)$ . This assumption holds, for instance, in the quadratic specification  $g(x) = \alpha x^2 + (1 - \alpha)x$ , where  $m = \frac{1}{2}$  regardless of  $\alpha \in (0, 1]$ . As shown by GMSZ-CRA, the quadratic model is consistent with many important models of non-expected utility, such as versions of the disappointment-aversion theories of Loomes and Sugden [1986] and Jia et al. [2001], Kőszegi and Rabin [2006]’s loss-averse utility, and modified mean-variance preferences [Blavatskyy, 2010].<sup>30</sup> Under this assumption, the direction of discrimination becomes monotone in types as shown in the proposition below.

**Proposition 5.** *Consider the model as above, and assume that the environment is regular. If each  $H_i$  is supermodular and  $\frac{\partial}{\partial x} H_i(m, m^{\frac{1}{n-1}}) > 0$ , then  $x_1^*(u) \geq x_n^*(u)$  if and only if  $u \leq m^{\frac{1}{n-1}}$ .*

The proposition shows that the optimal mechanism tilts in favor of the more risk-averse bidder at low quantiles and in favor of the less risk-averse bidder at high quantiles. Intuitively, when types are low, allocating probability to risk-averse bidders generates relatively smaller information rents. For high types, the concern shifts toward allocative efficiency, and the mechanism favors bidders whose valuations respond more strongly to additional probability—namely, the less risk-averse ones. This reversal of priorities across the type distribution is a direct consequence of marginal revenue equalization along the optimal principal curve.

**Example 3.** *Consider the quadratic specification with  $\alpha = 1$  and the uniform type distribution so that  $g(x) = x^2$  and  $v(u) = u$ . To illustrate, fix  $n = 3$ .<sup>31</sup>*

*When  $k = 2$  (two risk-neutral bidders and one risk-averse bidder), the optimal reduced form satisfies*

$$x_1^*(u) = \frac{2u + 2u^2 - 1}{2u + 1/u} \cdot \mathbf{1}_{[1/2, 1]}(u), \quad x_n^*(u) = \frac{u}{2(1 - u)} \cdot \mathbf{1}_{[0, 1/3]}(u) + \frac{2 - u^2 - \sqrt{3 - 2u^2}}{2(1 - u)^2} \cdot \mathbf{1}_{[1/3, 1]}(u).$$

*Here,  $T$  is finite, and low types of the risk-averse bidder receive their (unconstrained) optimal interim winning probability,  $x_n^*(u) = \frac{u}{2(1-u)}$ , as predicted by Proposition 5.*

*When  $k = 1$  (one risk-neutral bidder and two risk-averse bidders), the optimal reduced form satisfies*

$$x_1^*(u) = \frac{12u^2 - 8u + (2u - 1)^{3/2}\sqrt{10u - 1} + 1}{8u^2} \cdot \mathbf{1}_{[1/2, 1]}(u), \quad x_n^*(u) = \frac{u + 1}{2(1 - u) + 2/u}.$$

*In contrast to the previous case,  $T$  is infinite and the optimum is extremal.*

*In both cases, the tipping point in Proposition 5 equals  $m^{\frac{1}{2}} = \frac{1}{\sqrt{2}} \approx 0.707$ , and risk-averse bidders are favored for  $u < \frac{1}{\sqrt{2}}$  and disadvantaged for  $u > \frac{1}{\sqrt{2}}$ .*

## 10 Final remarks

In this paper, we develop new tools to study asymmetric auction settings with unidimensional types, unit demands, and a single good for sale.

<sup>30</sup>The unimodality condition also holds for  $g(x) = \frac{x}{1 + \alpha(1-x)}$ , which corresponds to Gul’s disappointment-averse preferences [Gul, 1991] with linear utility over outcomes.

<sup>31</sup>It is routine to verify that the environment is regular for all  $n, k$  (see the appendix for the discussion of regularity in terms of primitives) and that the added assumptions in Proposition 5 are satisfied.

We believe that these tools extend well beyond our specific framework and can help analyze other economically significant problems. One immediate application is information design under privacy constraints in the spirit of [He et al. \[2025\]](#), who establishes a connection between such information structures and Border-type feasibility in auction settings. Other potential applications include persuasion, delegation, and related problems studied in [Kleiner et al. \[2021\]](#) through the lens of symmetric Border-type feasibility.

Extending our approach to feasibility and extremality to more general auction settings is another fruitful direction for future research. Concretely, our method for identifying the principal curve of tightest feasibility constraints delivers a sharp characterization of feasibility and an explicit description of extremality. It may extend to settings with multiunit demands or even multidimensional types. Border-type feasibility characterizations are known in some of these environments (e.g., see [Che et al. \[2013\]](#)), and our approach may help make them more tractable and operational.

Our approach to optimality via optimal control may also prove valuable in broader principal–agent models and in persuasion. Even within our auction framework, we focus on regular environments in which ironing is not needed, and we illustrate how the results apply to settings with endogenous valuations and non-EU preferences. This focus is largely due to space constraints. Further developing the economics of optimal mechanisms in these applications and extending the analysis beyond regular environments are natural next steps.

## References

- P. Andreyanov, I. Krasikov, and A. Suzdaltsev. Scoring and favoritism in optimal procurement design. 2024.
- P. Andreyanov, I. Krasikov, and A. Suzdaltsev. Auctions after actions. 2026.
- I. Arieli, Y. Babichenko, R. Smorodinsky, and T. Yamashita. Optimal persuasion via bi-pooling. *Theoretical Economics*, 18(1):15–36, 2023.
- I. Ball. Bauer’s maximum principle for quasiconvex functions. *arXiv preprint arXiv:2305.04893*, 2023.
- P. R. Blavatsky. Modifying the mean-variance approach to avoid violations of stochastic dominance. *Management Science*, 56(11):2050–2057, 2010.
- K. C. Border. Implementation of reduced form auctions: A geometric approach. *Econometrica: Journal of the Econometric Society*, pages 1175–1187, 1991.
- K. C. Border. Reduced form auctions revisited. *Economic Theory*, 31(1):167–181, 2007.
- Y. Cai, C. Daskalakis, and S. M. Weinberg. A constructive approach to reduced-form auctions with applications to multi-item mechanism design. *arXiv preprint arXiv:1112.4572*, 2018.
- O. Candogan and P. Strack. Optimal disclosure of information to privately informed agents. *Theoretical Economics*, 18(3):1225–1269, 2023.
- G. Carroll and I. Segal. Robustly optimal auctions with unknown resale opportunities. *The Review of Economic Studies*, 86(4):1527–1555, 2019.
- Y.-K. Che, J. Kim, and K. Mierendorff. Generalized reduced-form auctions: A network-flow approach. *Econometrica*, 81(6):2487–2520, 2013.
- A. Gershkov, J. K. Goeree, A. Kushnir, B. Moldovanu, and X. Shi. On the equivalence of bayesian and dominant strategy implementation. *Econometrica*, 81(1):197–220, 2013.

- A. Gershkov, B. Moldovanu, P. Strack, and M. Zhang. A theory of auctions with endogenous valuations. *Journal of Political Economy*, 129(4):1011–1051, 2021.
- A. Gershkov, B. Moldovanu, P. Strack, and M. Zhang. Optimal auctions: Non-expected utility and constant risk aversion. *The review of economic studies*, 89(5):2630–2662, 2022.
- J. K. Goeree and A. Kushnir. A geometric approach to mechanism design. *Journal of Political Economy Microeconomics*, 1(2):321–347, 2023.
- F. Gul. A theory of disappointment aversion. *Econometrica: Journal of the Econometric Society*, pages 667–686, 1991.
- S. Hart and P. J. Reny. Implementation of reduced form mechanisms: a simple approach and a new characterization. *Economic Theory Bulletin*, 3(1):1–8, 2015.
- K. He, F. Sandomirskiy, and O. Tamuz. Private private information. *arXiv preprint arXiv:2112.14356*, 2025.
- J. Jia, J. S. Dyer, and J. C. Butler. Generalized disappointment models. *Journal of Risk and Uncertainty*, 22(1):59–78, 2001.
- A. Kleiner, B. Moldovanu, and P. Strack. Extreme points and majorization: Economic applications. *Econometrica*, 89(4):1557–1593, 2021.
- A. Kleiner, B. Moldovanu, P. Strack, and M. Whitmeyer. The extreme points of fusions. *arXiv preprint arXiv:2409.10779*, 2024.
- B. Köszegi and M. Rabin. A model of reference-dependent preferences. *The Quarterly Journal of Economics*, 121(4):1133–1165, 2006.
- P. Lahr and A. Niemeier. Extreme points in multi-dimensional screening. *arXiv preprint arXiv:2412.00649*, 2024.
- G. Loomes and R. Sugden. Disappointment and dynamic consistency in choice under uncertainty. *The Review of Economic Studies*, 53(2):271–282, 1986.
- A. M. Manelli and D. R. Vincent. Multidimensional mechanism design: Revenue maximization and the multiple-good monopoly. *Journal of Economic theory*, 137(1):153–185, 2007.
- A. M. Manelli and D. R. Vincent. Bayesian and dominant-strategy implementation in the independent private-values model. *Econometrica*, 78(6):1905–1938, 2010.
- E. Maskin and J. Riley. Optimal auctions with risk averse buyers. *Econometrica: Journal of the Econometric Society*, pages 1473–1518, 1984.
- S. A. Matthews. On the implementability of reduced form auctions. *Econometrica: Journal of the Econometric Society*, pages 1519–1522, 1984.
- K. Mierendorff. Asymmetric reduced form auctions. *Economics Letters*, 110(1):41–44, 2011.
- D. Mishra and A. Quadir. Non-bossy single object auctions. *Economic Theory Bulletin*, 2:93–110, 2014.
- R. B. Myerson. Optimal auction design. *Mathematics of operations research*, 6(1):58–73, 1981.
- A. Nikzad. Constrained majorization: Applications in mechanism design. 2023.

- M. A. Satterthwaite and H. Sonnenschein. Strategy-proof allocation mechanisms at differentiable points. *The Review of Economic Studies*, 48(4):587–597, 1981.
- A. Seierstad and K. Sydsaeter. *Optimal control theory with economic applications*. Elsevier North-Holland, Inc., 1986.
- M. Shaked and J. G. Shanthikumar. *Stochastic orders*. Springer, 2007.
- F. Yang and K. H. Yang. Multidimensional monotonicity and economic applications. *arXiv preprint arXiv:2502.18876*, 2025.
- K. H. Yang and A. K. Zentefis. Monotone function intervals: Theory and applications. *American Economic Review*, 114(8):2239–2270, 2024.
- M. Zhang. Auctions vs. negotiations: Optimal selling mechanism with endogenous bidder values. *Negotiations: Optimal Selling Mechanism with Endogenous Bidder Values (August 30, 2017)*, 2017.

## A Appendix

### A.1 Feasibility

This section proves Theorem 1 and collects basic properties of the principal curve—such as continuity and monotonicity—that are used later. The proof proceeds in two steps. Lemma 1 establishes an integral identity for the  $\psi$ -transform that we use to obtain (9). Lemma 2 is key: it shows that the principal curve characterizes the tightest Border constraints by studying an auxiliary maximization problem. Combining these lemmas yields Theorem 1.

#### A.1.1 Preliminaries

**Lemma 1.** *Let  $x \in \mathcal{X}$ . Then, its  $\psi$ -transform satisfies*

$$\int_{\psi(\iota)}^1 x(u) du = \int_{\iota}^1 s d \ln \psi(s) \quad \forall \iota \in [0, 1].$$

*Proof.* Let  $D$  be the set of discontinuities of  $x$  excluding a possible atom at  $\psi(0)$ , and recollect that  $[\psi(0), 1] \setminus D$  can be rewritten as a countable union of ordered intervals of the form  $[a^l, b^l]$ . On each such interval,  $ux$  is strictly increasing and continuous, thus  $\psi$  is strictly increasing on the image of this interval. So, we obtain

$$\begin{aligned} \int_{\psi(\iota)}^1 x(u) du &= \sum_l \int_{a^l}^{b^l} \mathbf{1}_{(\psi(\iota), 1]}(u) x(u) du \\ &= \sum_l \int_{a^l}^{b^l} \mathbf{1}_{(\psi(\iota), 1]} \psi^{-1}(u) d \ln u \\ &= \sum_l \int_{\psi^{-1}(a^l)}^{\psi^{-1}(b^l)} \mathbf{1}_{(\iota, 1]}(s) s d \ln \psi(s) \\ &= \int_{\iota}^1 s d \ln \psi(s). \end{aligned}$$

The second line invokes the definition of  $\psi$  and its properties. Specifically, on any continuity interval  $[a^l, b^l] \subseteq [\psi(0), 1]$ , we have  $\psi^{-1}(u) = ux(u)$ , hence  $x(u) du = \psi^{-1}(u) d \ln u$ . The third line is due to the change of variables of integration, i.e.,  $u = \psi(s)$ . Finally, the last line follows from the fact that  $\psi$  is constant on the complement of intervals of the form  $[\psi^{-1}(a^l), \psi^{-1}(b^l)]$ .  $\square$

**Lemma 2.** *Let  $\mathbf{x}$  be a reduced form. Denote its  $\psi$ -transforms by  $\psi$ , and set  $\bar{\psi} = \sqrt[n]{\prod_{i=1}^n \psi_i}$ . Consider the following maximization problem parameterized by  $s \in [0, 1]$ :*

$$G(s) = \max_{\mathbf{u} \in [0, 1]^n} s^n + \sum_{i=1}^n \int_{u_i}^1 x_i(u) du \quad \text{s.t.} \quad \prod_{i=1}^n u_i \geq s^n. \quad (34)$$

*Then:*

1. *The value of the principal curve at  $s$ , which is given by  $v_i(s) = \psi_i(\bar{\psi}^{-1}(s))$ , is optimal in (34) and uniquely optimal whenever  $s \in [\bar{\psi}(0), 1]$ ,  $s > 0$ .*
2. *The principal curve itself is continuous, constant on  $[0, \bar{\psi}(0))$  and strictly increasing in at least one coordinate on the complement of that set.*

*Proof.* To begin, remark that, for each  $s \in (0, 1)$ , the problem determining  $G$  is concave and can be solved through the following necessary (and sufficient due to concavity) first-order conditions:

$$\lambda \prod_{j=1}^n u_j \in [x_i(u_i -)u_i, x_i(u_i)u_i],$$

where  $\lambda$  is the Lagrange multiplier on the constraint.

If  $\lambda > 0$ , then the constraint binds, so  $s^n = \prod_{i=1}^n u_i$ . It follows that the first-order conditions can be rewritten as

$$u_i = \psi_i(\lambda s^n).$$

Taking the geometric mean across bidders gives  $s = \bar{\psi}(\lambda s^n)$ , and therefore  $s \geq \bar{\psi}(0)$  due to monotonicity of  $\bar{\psi}$ .

On the other hand, if  $\lambda = 0$ , the first-order conditions require

$$u_i \in [0, \psi_i(0)],$$

which implies  $s \leq \bar{\psi}(0)$ .

Equipped with this description of the first-order conditions, we now prove the first part of the lemma distinguishing between various values of the parameter  $s$ .

*Case  $s \in (\bar{\psi}(0), 1)$ .* As shown above, we must have  $\lambda > 0$  so that  $s = \bar{\psi}(\lambda s^n)$ . The largest Lagrange multiplier with this property can be obtained by taking the generalized inverse of  $\bar{\psi}$ , that is  $\lambda s^n = \bar{\psi}^{-1}(s)$ , which implies that

$$\nu_i(s) = \psi_i(\bar{\psi}^{-1}(s))$$

is a solution to the maximization problem that defines  $G$  for  $s > \bar{\psi}(0)$ . Since each  $\psi_i$  is constant on every interval on which  $\bar{\psi}$  is constant, if  $\tilde{\lambda} \in (0, \lambda)$  so that  $s = \bar{\psi}(\tilde{\lambda} s^n)$ , then  $\nu_i(s) = \psi_i(\tilde{\lambda} s^n)$ , which shows that  $\nu(s)$  is a unique maximizer.

*Case  $s \in (0, \bar{\psi}(0)]$ .* By construction,  $\nu_i(s) = \psi_i(0)$  satisfies the first-order conditions with  $\lambda = 0$ , thus it is optimal. Clearly, this point is unique one with that property for  $s = \bar{\psi}(0)$ .

*Case  $s = 1$ .* Remark that the only feasible point in the auxiliary problem is

$$\nu(1) = (1, \dots, 1),$$

and thus it is uniquely optimal.

*Case  $s = 0$ .* Note that the constraint in the auxiliary problem has no bite. By construction,  $x_i = 0$  on  $[0, \psi_i(0))$ , as a result

$$\nu(0) = (\psi_1(0), \dots, \psi_n(0))$$

maximizes the objective in (34) when the constraint is ignored.

As for the second part of the lemma, both the continuity and constancy of the principal curve on  $[0, \bar{\psi}(0))$  directly follow from its definition and the description above. Finally, take  $\underline{s} < \bar{s}$  in  $[\bar{\psi}(0), 1]$ . By construction, the principal curve is weakly increasing, and thus  $\nu(\underline{s}) \leq \nu(\bar{s})$ . Using the fact the constraint in the auxiliary problem holds as equality at both points, we obtain

$$\prod_{i=1}^n \nu_i(\underline{s}) = \underline{s}^n < \bar{s}^n = \prod_{i=1}^n \nu_i(\bar{s}),$$



which shows that  $\nu_i(\underline{s}) < \nu_i(\bar{s})$  for at least one coordinate establishing strict monotonicity of the principal curve on  $[\bar{\psi}(0), 1]$ .  $\square$

### A.1.2 Proof of Theorem 1

*Proof.* In (34),  $s^n$  is fixed in the objective, so decreasing  $u_i$  increases  $\int_{u_i}^1 x_i(u)du$  due to monotonicity of  $x_i$ . As a result, an optimizer can be taken with the product constraint binding, which gives

$$G(s) = \max_{\mathbf{u} \in [0,1]^n} B(\mathbf{u}) \text{ s.t. } \prod_{i=1}^n u_i = s^n$$

for each  $s \in [0, 1]$ . As a result, a reduced form  $\mathbf{x}$  is feasible if and only if the function  $G$  never exceeds 1. As explained in Lemma 2,  $G$  is strictly increasing on  $[0, \bar{\psi}(0))$  and the value of Border's constraint along the principal curve satisfies

$$B(\boldsymbol{\nu}(s)) = G\left(\max\{s, \bar{\psi}(0)\}\right). \quad (35)$$

Taking these observations together, we conclude that a reduced form  $\mathbf{x}$  is feasible if and only if the function  $B$  never exceeds 1 along the principal curve.

To see the second part of the theorem, remark that

$$\begin{aligned} \sum_{i=1}^n \int_{\nu_i(s)}^1 x_i(u)du &= \sum_{i=1}^n \int_{\bar{\psi}^{-1}(s)}^1 \iota d \ln \psi_i(\iota) \\ &= n \int_{\bar{\psi}^{-1}(s)}^1 \iota d \ln \bar{\psi}(\iota), \end{aligned}$$

where the first line is due to the definition of the principal curve and Lemma 1, whereas the second follows from the definition of  $\bar{\psi}$  as the geometric average of individual  $\psi$ -transforms. Then, Equation (9) ensues from combining this integral representation of  $\sum_{i=1}^n \int_{\nu_i(s)}^1 x_i(u)du$  with  $\prod_{i=1}^n \nu_i(s) = \left(\max\{s, \bar{\psi}(0)\}\right)^n$ .  $\square$

## A.2 Extremality

In this section, we study extremality of reduced forms and prove Theorems 2, 3. We show two theorems in conjunction to each other establishing three statements in the following order:

*Statement I.* If a feasible reduced form  $\mathbf{x}$  is extremal, then Border's constraint binds along the principal curve, or equivalently (II) is verified.

*Statement II.* If a feasible reduced form  $\mathbf{x}$  satisfies (II), then it can be induced by scores as described in Theorem 3.

*Statement III.* If a feasible reduced form  $\mathbf{x}$  can be induced by scores as described in Theorem 3, then it is extremal.

Statement I corresponds to the "only if" direction of Theorem 2, and the other two statements establish Theorem 3 provided that Statement I had been shown. Taken together, I-III imply the "if" direction of Theorem 2.

The most challenging part of the analysis is the first statement, which says Border's constraint binds along the principal curve for extremal reduced forms. In view of Lemma 2 and the proof of Theorem 1, this is equivalent to

the function  $G$ , which is defined in (3.4), to satisfy

$$G(s) = 1 \quad \forall s \in [\bar{\psi}(0), 1].$$

This condition is equivalent to non-existence of a *sag*, that is an open interval  $(\underline{s}, \bar{s})$  in  $[\bar{\psi}(0), 1]$  on which  $G$  is strictly less than 1.

For each feasible (potentially non-extremal) reduced form  $\mathbf{x}$ , if the set of sags is non-empty, then, since  $G(1) = 1$  and  $G < 1$  on  $[0, \bar{\psi}(0))$ , there are two mutually exclusive and exhaustive cases:

1. There exists a sag so that  $G$  equals 1 at its endpoints.
2. There exists a sag that starts at  $\bar{\psi}(0)$  and ends at  $\bar{s}$  so that  $G$  equals 1 on  $[\bar{s}, 1]$ .

We shall show both cases are inconsistent with extremality of  $\mathbf{x}$ , thereby establishing that no extremal reduced form admits a sag. To this end, we first state and prove six auxiliary lemmas that record various properties of elements of the feasible set and its extreme points. With these tools in hand, we then prove the two theorems stated in Section 6.

### A.2.1 Preliminaries

Lemmas 3 and 4 show that certain modifications of a feasible reduced form leave (some of) Border's constraints unchanged. For brevity, when we need to emphasize the dependence on the reduced form, we write  $B_{\mathbf{x}}$  for Border's constraint in (6) corresponding to  $\mathbf{x}$ .

The other three lemmas in this section study sags of extremal reduced forms. Specifically, Lemma 5 establishes that  $x_i$  is piecewise-constant with finitely many jumps along a sag of the principal curve, and Lemmas 6, 7 discuss jumps at the endpoints of the sag. Finally, Lemma 8 shows that any sag corresponds to a non-trivial portion of the principal curve. Throughout this analysis, we use the following notation. Given a reduced form  $\mathbf{x}$ , and two cutoffs  $\underline{u} < \bar{u}$ , define the set of jumps of  $x_i$  in  $[\underline{u}, \bar{u}]$  by

$$D_i(\underline{u}, \bar{u}) = \{u \in [\underline{u}, \bar{u}] : x_i(u-) \neq x_i(u)\}$$

with convention  $x_i(0-) = 0$ . We extend the definition of  $D_i$  to degenerate intervals by setting

$$D_i(u, u) = \{u\}.$$

**Lemma 3** (Perturbations at the bottom of the type space). *Let  $\mathbf{x}$  be a feasible reduced form that satisfies  $B_{\mathbf{x}}(\bar{\mathbf{u}}) = 1$  for some  $\bar{\mathbf{u}} \in [0, 1]^n$ ,  $\prod_{i=1}^n \bar{u}_i > 0$ . Consider  $\Delta = (\Delta_1, \dots, \Delta_n)$  with  $\Delta_i : [0, 1] \rightarrow \mathbf{R}$  such that  $\mathbf{x} + \Delta$  is a reduced form,  $\Delta \neq 0$ ,  $\Delta_i(u) = 0$  on  $(\bar{u}_i, 1]$  for each  $i$ . Then,  $\mathbf{x} + \Delta$  is feasible if and only if*

$$B_{\mathbf{x}+\Delta}(\mathbf{u}) \leq 1 \quad \forall \mathbf{u} \text{ s.t. } \mathbf{0} \leq \mathbf{u} \leq \bar{\mathbf{u}}. \quad (36)$$

*Proof.* The “only if” direction is immediate, so we focus on the converse. Specifically, suppose that (36) holds. We now show that  $\mathbf{x} + \Delta$  is feasible, meaning that the inequality in (36) holds for all  $\mathbf{u}$  other than  $\mathbf{0} \leq \mathbf{u} \leq \bar{\mathbf{u}}$ . To this end, we distinguish between several cases.

*Case*  $\prod_{i=1}^n u_i = 0$ . In this case,

$$B_{\mathbf{x}+\Delta}(\mathbf{u}) \leq \sum_{i=1}^n \int_0^1 (x_i(u) + \Delta_i(u)) du = B_{\mathbf{x}+\Delta}(\mathbf{0}) \leq 1,$$

where the first inequality holds because  $\prod_{i=1}^n u_i = 0$ ,  $x_i + \Delta_i \geq 0$ , and the last inequality follows from (36).

Conversely, suppose  $\mathbf{u} \in [0, 1]^n$  is such that  $\prod_{i=1}^n u_i > 0$  and  $u_i > \bar{u}_i$  precisely for the first  $k \neq 0$  coordinates, i.e.,  $i = 1, \dots, k$ —after relabeling if necessary.

*Case  $k = n$ .* Since  $\mathbf{u} \geq \bar{\mathbf{u}}$ , we have  $\int_{u_i}^1 \Delta_i(u) du = 0$  for each  $i \in N$ , and hence

$$B_{\mathbf{x}+\Delta}(\mathbf{u}) = B_{\mathbf{x}}(\mathbf{u}) \leq 1.$$

*Case  $k < n$ .* Set  $a_1 = \prod_{i=1}^k u_i$ ,  $\bar{a}_1 = \prod_{i=1}^k \bar{u}_i$ ,  $a_2 = \prod_{i=k+1}^n u_i$ , and  $\bar{a}_2 = \prod_{i=k+1}^n \bar{u}_i$ .

Since  $\mathbf{x}$  is feasible and  $B_{\mathbf{x}}(\bar{\mathbf{u}}) = 1$ , we have

$$B_{\mathbf{x}}(u_1, \dots, u_k, \bar{u}_{k+1}, \dots, \bar{u}_n) \leq B_{\mathbf{x}}(\bar{\mathbf{u}}),$$

which expands to

$$a_1 \cdot \bar{a}_2 + \sum_{i=1}^k \int_{u_i}^1 x_i(u) du \leq \bar{a}_1 \cdot \bar{a}_2 + \sum_{i=1}^k \int_{\bar{u}_i}^1 x_i(u) du. \quad (37)$$

Next, consider evaluating  $B_{\mathbf{x}+\Delta}$  at  $(\bar{u}_1, \dots, \bar{u}_k, u_{k+1}, \dots, u_n)$ . Using  $\Delta_i(u) = 0$  for  $u > \bar{u}_i$ , we obtain

$$\bar{a}_1 \cdot a_2 + \sum_{i=1}^k \int_{\bar{u}_i}^1 x_i(u) du + \sum_{i=k+1}^n \int_{u_i}^1 x_i(u) du + \sum_{i=1}^n \int_{u_i}^1 \Delta_i(u) du \leq 1, \quad (38)$$

due to (36).

Since  $a_1 \geq \bar{a}_1$  and  $a_2 \leq \bar{a}_2$ , we have  $(a_1 - \bar{a}_1) \cdot (a_2 - \bar{a}_2) \leq 0$ , i.e.,

$$a_1 \cdot a_2 \leq \bar{a}_1 \cdot a_2 + a_1 \cdot \bar{a}_2 - \bar{a}_1 \cdot \bar{a}_2.$$

Combining this inequality with (37) and (38), we conclude that

$$\begin{aligned} B_{\mathbf{x}+\Delta}(\mathbf{u}) &= a_1 \cdot a_2 + \sum_{i=1}^k \int_{u_i}^1 x_i(u) du + \sum_{i=1}^n \int_{u_i}^1 \Delta_i(u) du \\ &\leq \bar{a}_1 \cdot a_2 + a_1 \cdot \bar{a}_2 - \bar{a}_1 \cdot \bar{a}_2 + \sum_{i=1}^k \int_{u_i}^1 x_i(u) du + \sum_{i=1}^n \int_{u_i}^1 \Delta_i(u) du \\ &\leq \bar{a}_1 \cdot a_2 + \sum_{i=1}^k \int_{\bar{u}_i}^1 x_i(u) du + \sum_{i=k+1}^n \int_{u_i}^1 x_i(u) du + \sum_{i=1}^n \int_{u_i}^1 \Delta_i(u) du \\ &\leq 1, \end{aligned}$$

as claimed.  $\square$

where the first line follows from the definitions of  $B_{\mathbf{x}+\Delta}$ , the second one uses

**Lemma 4** (Perturbations in the middle of the type space). *Let  $\mathbf{x}$  be a feasible reduced form that satisfies  $B_{\mathbf{x}}(\underline{\mathbf{u}}) = B_{\mathbf{x}}(\bar{\mathbf{u}}) = 1$  for some  $\underline{\mathbf{u}}, \bar{\mathbf{u}} \in [0, 1]^n$  such that  $\prod_{i=1}^n \underline{u}_i > 0$ ,  $\underline{\mathbf{u}} \neq \bar{\mathbf{u}}$ ,  $\underline{\mathbf{u}} \leq \bar{\mathbf{u}}$ . Consider  $\Delta = (\Delta_1, \dots, \Delta_n)$  with  $\Delta_i : [0, 1] \rightarrow \mathbf{R}$  such that  $\mathbf{x} + \Delta$  is a reduced form,  $\Delta \neq 0$ ,  $\Delta_i(u) = 0$  on  $[0, \underline{u}_i) \cup (\bar{u}_i, 1]$  for each  $i$ , and*

$$\sum_{i=1}^n \int_{\underline{u}_i}^{\bar{u}_i} \Delta_i(u) du = 0.$$

Then,  $\mathbf{x} + \Delta$  is feasible if and only if

$$B_{\mathbf{x}+\Delta}(\mathbf{u}) \leq 1 \quad \forall \mathbf{u} \text{ s.t. } \underline{\mathbf{u}} \leq \mathbf{u} \leq \bar{\mathbf{u}}. \quad (39)$$

*Proof.* In view of Lemma 3, the reduced form  $\mathbf{x} + \Delta$  is feasible if and only if (36) holds. We thus need to show that (36) is satisfied if and only if (39) holds. The "only if" direction is immediate, so we focus on the converse, again distinguishing between several cases.

*Case*  $\prod_{i=1}^n u_i = 0$ . In this case,

$$B_{\mathbf{x}+\Delta}(\mathbf{u}) \leq \sum_{i=1}^n \int_0^1 (x_i(u) + \Delta_i(u)) du = B_{\mathbf{x}}(\mathbf{0}) \leq 1,$$

where the first inequality holds because  $\prod_{i=1}^n u_i = 0, x_i + \Delta_i \geq 0$ , the equality follows from  $\sum_{i=1}^n \int_0^1 \Delta_i(u) du = 0$ , and the last inequality follows from feasibility of  $\mathbf{x}$ .

Now, suppose  $\mathbf{u} \leq \bar{\mathbf{u}}$  is such that  $\prod_{i=1}^n u_i > 0$  and  $u_i < \underline{u}_i$  precisely for the first  $k \neq 0$  coordinates, i.e.,  $i = 1, \dots, k$ —after relabeling if necessary.

*Case*  $k = n$ . Since  $\mathbf{u} \leq \underline{\mathbf{u}}$ , it follows that  $\sum_{i=1}^k \int_{u_i}^1 \Delta_i(u) du = 0$ , and hence

$$B_{\mathbf{x}+\Delta}(\mathbf{u}) = B_{\mathbf{x}}(\mathbf{u}) \leq 1.$$

*Case*  $k < n$ . The argument is identical to the one in the proof of Lemma 3 but replaces  $\bar{\mathbf{u}}$  with  $\underline{\mathbf{u}}$ . Setting  $a_1 = \prod_{i=1}^k u_i, \underline{a}_1 = \prod_{i=1}^k \underline{u}_i, a_2 = \prod_{i=k+1}^n u_i, \underline{a}_2 = \prod_{i=k+1}^n \underline{u}_i$ , and repeating the same steps as before, we can obtain

$$\begin{aligned} a_1 \cdot \underline{a}_2 + \sum_{i=1}^k \int_{u_i}^1 x_i(u) du &\leq \underline{a}_1 \cdot \underline{a}_2 + \sum_{i=1}^k \int_{\underline{u}_i}^1 x_i(u) du, \\ \underline{a}_1 \cdot a_2 + \sum_{i=1}^k \int_{\underline{u}_i}^1 x_i(u) du + \sum_{i=k+1}^n \int_{u_i}^1 x_i(u) du + \sum_{i=1}^n \int_{u_i}^1 \Delta_i(u) du &\leq 1, \\ a_1 \cdot a_2 &\leq a_1 \cdot \underline{a}_2 + \underline{a}_1 \cdot a_2 - \underline{a}_1 \cdot \underline{a}_2. \end{aligned}$$

Combining these inequalities yields the desired result:

$$B_{\mathbf{x}+\Delta}(\mathbf{u}) \leq 1.$$

□

**Lemma 5** (Structure along a sag). *Let  $\mathbf{x}$  be an extremal reduced form. Consider a sag  $(\underline{s}, \bar{s})$ . Then, for each  $i$  such that  $\nu_i(\underline{s}) < \nu_i(\bar{s})$ , the function  $x_i$  is piecewise-constant on  $[\nu_i(\underline{s}), \nu_i(\bar{s})]$  with finitely elements in  $D_i(\nu_i(\underline{s}), \nu_i(\bar{s}))$ , i.e.,  $x_i$  takes finitely many values.*

*Proof.* Let  $i$  be such that  $\nu_i(\underline{s}) < \nu_i(\bar{s})$ . There are two mutually exclusive possibilities: either all (weak) lower contour sets of  $x_i$  are open in  $(\nu_i(\underline{s}), \nu_i(\bar{s}))$ , i.e.,

$$\{u \in (\nu_i(\underline{s}), \nu_i(\bar{s})) \mid x(u) \leq x(\underline{u})\} \text{ is open } \forall \underline{u} \in (\nu_i(\underline{s}), \nu_i(\bar{s})), \quad (40)$$

or there exists at least one such lower contour set that is not open. In what follows, we first prove that the second case cannot occur given that  $\mathbf{x}$  is extremal in two logical steps, and then complete the proof in one additional step.

*Step 1.* Towards a contradiction, suppose that (40) is violated at  $\underline{u}$ , i.e., the associated lower contour set equals  $(\nu_i(\underline{s}), \underline{u}]$ . This means that there exists a closed interval  $[\underline{u}, \bar{u}]$  compactly contained in  $(\nu_i(\underline{s}), \nu_i(\bar{s}))$  such that  $x_i(u) > x_i(\underline{u})$  for all  $u \in (\underline{u}, \bar{u}]$ . Denote the value in

$$\max_{\mathbf{u} \in [0,1]^n} B(\mathbf{u}) \quad \text{s.t. } u_i \in [\underline{u}, \bar{u}].$$

by  $\omega$ . Since  $\mathbf{x}$  is feasible, we have  $\omega \leq 1$ . We now show that  $\omega$  is strictly less than one.

Suppose, for contradiction, that  $\omega = 1$ . Let  $\mathbf{u}$  a maximizer that attains  $\omega$  and set  $s \in [0, 1]$  so that  $s^n = \prod_{i=1}^n u_i$ . By definition of the auxiliary problem in (2),  $G(s)$  is an upper bound on  $\omega$ . This bound is tight because  $\mathbf{x}$  is assumed to be feasible, i.e.,  $G \leq 1$ , and  $\omega = 1$ . Since  $G$  is strictly less than one on  $[0, \bar{\psi}(0))$ , we must have  $s \geq \bar{\psi}(0)$ , thus

$$B(\mathbf{u}) = G(s) = B(\boldsymbol{\nu}(s))$$

due to (35) in the proof of Theorem 1.

Since  $G$  is strictly less than one on  $(\underline{s}, \bar{s})$ , we cannot have  $s \in (\underline{s}, \bar{s})$ . By monotonicity of the principal curve,

$$\nu_i(s) \leq \nu_i(\underline{s}) < \underline{u}$$

whenever  $s \leq \underline{s}$ , and

$$\nu_i(s) \geq \nu_i(\bar{s}) > \bar{u}$$

whenever  $s \geq \bar{s}$ . In either case, we obtain  $\nu_i(s) \notin [\underline{u}, \bar{u}]$ . As a result,  $\mathbf{u} \neq \boldsymbol{\nu}(s)$  contradicting the uniqueness of the optimizer stated in Lemma 2. Therefore, we conclude that  $\omega < 1$ , as claimed.

*Step 2.* We now construct a perturbation  $\boldsymbol{\Delta} : [0, 1] \rightarrow \mathbf{R}^n$  so that  $\mathbf{x} \pm \varepsilon \boldsymbol{\Delta}$  is a feasible reduced form provided that  $\varepsilon > 0$  is small enough. The existence of such perturbation contradicts extremality of  $\mathbf{x}$ , thereby ruling out the existence of a non-open lower contour set.

First of all, for each  $j \neq i$ , set  $\Delta_j = 0$ . As shown in the proof of Theorem 1 in Kleiner et al. [2021], there exists a number  $x^* \in [x_i(\underline{u}), x_i(\bar{u})]$  and  $[\underline{u}^*, \bar{u}^*]$  contained in that interval so that  $\Delta_i$  defined by

$$\Delta_i(u) = \begin{cases} x_i(u) - x_i(\underline{u}) & u \in [\underline{u}, \underline{u}^*), \\ x^* - x_i(u) & u \in [\underline{u}^*, \bar{u}^*), \\ x_i(u) - x_i(\bar{u}) & u \in [\bar{u}^*, \bar{u}], \\ 0 & u \notin [\underline{u}, \bar{u}] \end{cases}$$

is such that  $\Delta_i \neq 0$ ,  $x_i \pm \Delta_i$  is in  $\mathcal{X}$ , and  $\int_{\underline{u}}^{\bar{u}} \Delta_i(u) du = 0$ . In fact, Kleiner et al. [2021] provided a closed-form expression for the endpoints of the subinterval used above:

$$\begin{aligned} \underline{u}^* &= \inf_{u \in [\underline{u}, \bar{u}]} u \quad \text{s.t. } 2x_i(u) \geq x_i(\underline{u}) + x^*, \\ \bar{u}^* &= \inf_{u \in [\underline{u}, \bar{u}]} u \quad \text{s.t. } 2x_i(u) \geq x_i(\bar{u}) + x^* \end{aligned}$$

and proved that the value of  $x^*$  making  $\int_{\underline{u}}^{\bar{u}} \Delta_i(u) du = 0$  (provided the endpoints chosen as above) is well-defined defined.

Set

$$\bar{\varepsilon} = \frac{1 - \omega}{\max_{u_i \in [\underline{u}, \bar{u}]} \left| \int_{u_i}^{\bar{u}} \Delta_i(u) du \right|} > 0$$

and choose  $\varepsilon \in (0, 1)$  such that  $\varepsilon \leq \bar{\varepsilon}$ . Then, for every  $\mathbf{u} \in [0, 1]^n$  with  $u_i \in [\underline{u}, \bar{u}]$ , we have

$$B_{\mathbf{x} \pm \varepsilon \Delta}(\mathbf{u}) \leq B_{\mathbf{x}}(\mathbf{u}) + \varepsilon \left| \int_{u_i}^1 \Delta_i(u) du \right| \leq B_{\mathbf{x}}(\mathbf{u}) + (1 - \omega).$$

Since  $B_{\mathbf{x}}(\mathbf{u}) \leq \omega$  for every  $\mathbf{u} \in [0, 1]^n$  with  $u_i \in [\underline{u}, \bar{u}]$  and  $B_{\mathbf{x} \pm \varepsilon \Delta}(\mathbf{u}) = B_{\mathbf{x}}(\mathbf{u}) \leq 1$  for every  $\mathbf{u} \in [0, 1]^n$  with  $u_i \notin [\underline{u}, \bar{u}]$ , the reduced form  $\mathbf{x} \pm \varepsilon \Delta$  is feasible. However, since  $\mathbf{x}$  is assumed to be an extreme point of the feasible set, this contradicts extremality. Therefore, no such interval  $[\underline{u}, \bar{u}]$  can exist.

To sum up, we have shown all (weak) lower contours sets of  $x_i$  are open in  $(\nu_i(\underline{s}), \nu_i(\bar{s}))$ . This means that  $x_i$  is piecewise-constant and its set of jumps  $D_i(\nu_i(\underline{s}), \nu_i(\bar{s}))$  is well-ordered in reverse, i.e., every its nonempty subset admits the largest element, and so we can meaningfully talk about consecutive jumps in that set.

*Step 3.* To conclude the proof, we need to show that it is finite. By the way of contradiction, suppose  $x_i$  admits three consecutive jumps points in  $(\nu_i(\underline{s}), \nu_i(\bar{s}))$  denoted  $\underline{u} < u^* < \bar{u}$ . Similarly to the previous step, we rule out this possibility by explicitly constructing a perturbation  $\Delta = (\Delta_1, \dots, \Delta_n) : [0, 1] \rightarrow \mathbf{R}^n$  so that  $\mathbf{x} \pm \varepsilon \Delta$  is a feasible reduced form provided that  $\varepsilon > 0$  is small enough.

First of all, for each  $j \neq i$ , set  $\Delta_j = 0$ . For coordinate  $i$ , set

$$\Delta_i(u) = \kappa \cdot \left( \frac{\mathbf{1}_{[\underline{u}, u^*)}(u)}{u^* - \underline{u}} - \frac{\mathbf{1}_{[u^*, \bar{u})}(u)}{\bar{u} - u^*} \right) \quad (41)$$

for some  $\kappa > 0$  small enough to ensure  $\Delta_i \neq 0$  and  $x_i \pm \Delta_i \in \mathcal{X}$ , e.g., any choice such that

$$\frac{\kappa}{u^* - \underline{u}} \leq x_i(\underline{u}) - x_i(\underline{u}-), \quad \frac{\kappa}{u^* - \underline{u}} + \frac{\kappa}{\bar{u} - u^*} \leq x_i(u^*) - x_i(u^*-), \quad \frac{\kappa}{\bar{u} - u^*} \leq x_i(\bar{u}) - x_i(\bar{u}-)$$

will do.

Then, applying the same argument as in the previous construction, we conclude that the perturbed reduced form  $\mathbf{x} \pm \varepsilon \Delta$  is feasible for sufficiently small  $\varepsilon > 0$ . This contradicts extremality of  $\mathbf{x}$ , and hence no such triple jump structure can exist.  $\square$

**Lemma 6** (Discontinuity at the top of a sag). *Let  $\mathbf{x}$  be an extremal reduced form. Consider a sag  $(\underline{s}, \bar{s})$  satisfying  $G(\bar{s}) = 1$ . Then,*

$$\nu_i(\bar{s}) \in D_i(\nu_i(\underline{s}), \nu_i(\bar{s})) \quad \forall i,$$

*i.e.,  $x_i$  is discontinuous at  $\nu_i(\bar{s})$  whenever  $\nu_i(\underline{s}) < \nu_i(\bar{s})$ .*

*Proof.* Fix an index  $i$  with  $\nu_i(\underline{s}) < \nu_i(\bar{s})$ , and suppose, towards a contradiction, that  $x_i$  is continuous at  $\nu_i(\bar{s})$ . We distinguish between two cases.

*Case  $\nu_i(\bar{s}) < 1$ .* Clearly, we have  $\bar{s} < 1$ . For  $s$  near  $\bar{s}$ , define  $\mathbf{u}(s)$  by keeping all coordinates except  $i$  fixed at their values on the principal curve at  $\bar{s}$ , that is  $u_j(s) = \nu_j(\bar{s})$  for  $j \neq i$ , and adjusting coordinate  $i$  to satisfy the product constraint:

$$u_i(s) = \frac{\nu_i(\bar{s})}{\bar{s}^n} s^n.$$

By construction,  $\prod_{j=1}^n u_j(s) = s^n$ , so  $\mathbf{u}(s)$  is feasible in the optimization problem that defines  $G(s)$  in Lemma

2. Hence,  $G(s) \geq \underline{G}(s)$ , where  $\underline{G}(s)$  is the objective evaluated at  $\mathbf{u}(s)$ :

$$\underline{G}(s) = s^n + \sum_{j=1}^n \int_{u_j(s)}^1 x_j(u) du = s^n + \int_{\frac{\nu_i(\bar{s})}{s^n} s^n}^1 x_i(u) du + \sum_{j \neq i} \int_{\nu_j(\bar{s})}^1 x_j(u) du. \quad (42)$$

On the one hand,  $\underline{G}(\bar{s}) = G(\bar{s}) = 1$ . On the other hand, since  $(s, \bar{s})$  is a sag, we have  $\underline{G}(s) \leq G(s) < 1$  for  $s < \bar{s}$ .

By Lemma 5, the function  $x_i$  is piecewise-constant on  $[\nu_i(s), \nu_i(\bar{s})]$  with finitely many jumps. Since,  $x_i$  is assumed to be continuous at the right end point, it is constant on a left neighborhood of this point. Consequently,  $\underline{G}$  is an affine function of  $s^n$  for  $s < \bar{s}$ . But  $\underline{G}(s) < \underline{G}(\bar{s})$  for  $s < \bar{s}$ , meaning that the (left) slope of this affine function is strictly positive, therefore  $\underline{G}(s) > 1$  for  $s > \bar{s}$ . This implies that  $G(s)$  strictly exceeds 1 contradicting  $\underline{G} \leq G \leq 1$  on a right neighborhood of  $\bar{s}$ , as required by feasibility. Thus, we conclude that  $x_i$  cannot be continuous at  $\nu_i(\bar{s})$ .

*Case  $\nu_i(\bar{s}) = 1$ .* The argument is virtually identical to the previous case, with the only difference that now  $x_i(1) = 1$  and continuity at  $\nu_i(\bar{s}) = 1$  implies  $x_i(u) = 1$  on a left neighborhood of 1. Hence, for  $s < \bar{s}$  close enough to  $\bar{s}$ , the same construction of  $\mathbf{u}(s)$  and the same lower bound  $\underline{G}$  in (42) apply, and  $\underline{G}$  is affine in  $s^n$  on a left neighborhood of  $\bar{s}$ . Its (left) slope equals

$$\frac{\partial}{\partial s^n} \underline{G}(\bar{s}-) = 1 - \frac{\nu_i(\bar{s})}{s^n} x_i(\nu_i(\bar{s})) = 1 - \frac{1}{s^n} \leq 0.$$

Since  $\underline{G}(\bar{s}) = G(\bar{s}) = 1$ , linearity and the nonpositive slope imply that  $\underline{G}(s) \geq 1$  for all  $s < \bar{s}$  close enough to  $\bar{s}$  contradicting that  $(s, \bar{s})$  is a sag. Therefore,  $x_i$  must be discontinuous at  $\nu_i(\bar{s}) = 1$ .  $\square$

**Lemma 7** (Discontinuity at the bottom of a sag). *Let  $\mathbf{x}$  be an extremal reduced form. Consider a sag  $(s, \bar{s})$  satisfying  $G(s) = 1$  with  $s > 0$ . Then,*

$$\nu_i(s) \in D_i(\nu_i(s), \nu_i(\bar{s})) \quad \forall i,$$

*i.e.,  $x_i$  is discontinuous at  $\nu_i(s)$  whenever  $\nu_i(s) < \nu_i(\bar{s})$ .*

*Proof.* Let  $i$  be such that  $\nu_i(s) < \nu_i(\bar{s})$ . Since  $s > 0$ , it must be the case that  $\nu_i(s) > 0$ . The argument is identical to the first case in the proof of Lemma 6. Specifically, suppose, for contradiction, that  $x_i$  is continuous at  $\nu_i(s)$ . Then, the lower bound  $\underline{G}$  is well-defined in a neighborhood of  $s$ , and we have

$$\frac{\partial}{\partial s^n} \underline{G}(s-) = \frac{\partial}{\partial s^n} \underline{G}(s+) = 1 - \frac{\nu_i(s)}{s^n} x_i(\nu_i(s)) < 0,$$

where the equality follows from the continuity of  $x_i$  at  $\nu_i(s)$ , and the inequality follows from the fact that  $x_i$  is constant in a right neighborhood of  $\nu_i(s)$ , together with the assumption that  $\underline{G} \leq G < 1$  on  $(s, \bar{s})$ . But this implies that  $G \geq \underline{G} > 1$  on a left neighborhood of  $s$ , contradicting feasibility. Therefore,  $x_i$  is discontinuous at  $\nu_i(s)$ .  $\square$

**Lemma 8** (Non-triviality along a sag). *Let  $\mathbf{x}$  be a feasible reduced form. Consider a sag  $(s, \bar{s})$ . Then, we have (1)  $\nu_i(s) < \nu_i(\bar{s})$  for at least one index  $i$ ; and (2) it is true for at least two such indices whenever  $G$  equals 1 at the endpoints of the sag.*

*Proof.* The first part of this claim has been established in Lemma 2, and so we focus on the second part.

Suppose, for contradiction, that  $\nu_i(s) < \nu_i(\bar{s})$  but  $\nu_j(s) = \nu_j(\bar{s})$  for all  $j \neq i$ . For each such  $j \neq i$ , since  $u_j^*$

is increasing,  $\nu_j$  equals to  $\nu_j(\bar{s})$  on  $[\underline{s}, \bar{s}]$ . Therefore, for all  $s \in [\underline{s}, \bar{s}]$ , we have

$$\nu_i(s) = \frac{1}{\prod_{j \neq i} \nu_j(\bar{s})} s^n = \frac{\nu_i(\bar{s})}{\bar{s}^n} s^n.$$

It follows that  $G(s)$  equals  $\underline{G}(s)$  defined in (42) for all  $s \in [\underline{s}, \bar{s}]$ . Clearly,  $G$  is a concave function of  $s^n$  on the respective interval. Since  $G(\underline{s}) = G(\bar{s}) = 1$ , it must be that  $G = 1$  on  $(\underline{s}, \bar{s})$  due to concavity. This contradicts the premise of the claim, completing the proof by contradiction.  $\square$

### A.2.2 Proofs of Theorems 2 and 3: Statement I

*Proof.* Let  $\mathbf{x}$  be an extremal reduced form. As explained above, there are two mutually exclusive cases, namely I and II. We now analyze each in detail starting with the latter.

*Case 2 with  $\bar{s} > \bar{\psi}(0)$ .* Assume  $G < 1$  on  $[\bar{\psi}(0), \bar{s})$  and  $G = 1$  on  $[\bar{s}, 1]$  for some  $\bar{s} > \bar{\psi}(0)$ . As explained in Lemma 2, we also know that  $G$  is strictly increasing on  $[0, \bar{\psi}(0))$ , and thus  $G < 1$  on that interval.

To simplify notation, set  $\bar{\mathbf{u}} = \boldsymbol{\nu}(\bar{s})$  and  $\underline{\mathbf{u}} = \boldsymbol{\nu}(\bar{\psi}(0))$ . Since  $\bar{s} > 0$ , we have  $\prod_{i=1}^n \bar{u}_i > 0$ . Lemmas 5 and 6 jointly imply that for each  $i$  with  $\underline{u}_i < \bar{u}_i$ , the function  $x_i$  is piecewise-constant on this interval with finitely many jumps. Furthermore, the upper endpoint is a discontinuity point, i.e.,  $\bar{u}_i \in D_i(\underline{u}_i, \bar{u}_i)$ . Since  $x_i = 0$  on  $[0, \underline{u}_i)$ , by construction, it follows that the discontinuity points up to  $\bar{u}_i$  are the same whether one starts at  $\underline{u}_i$  or 0:

$$D_i(\underline{u}_i, \bar{u}_i) = D_i(0, \bar{u}_i).$$

*Step 1.* To begin, we show that there exists an index  $i$  for which the discontinuity set  $D_i(0, \bar{u}_i)$  contains at least two elements. Suppose, towards a contradiction, that

$$D_i(0, \bar{u}_i) = \{\bar{u}_i\} \quad \forall i.$$

This means that  $x_i = 0$  on  $[0, \bar{u}_i)$ . Since, by the definition of the transformation  $\psi_i$ , the function  $x_i(u) > 0$  if and only if  $u \geq \psi_i(0)$ , the upper endpoint  $\bar{u}_i$  is not lower than  $\psi_i(0)$  for all  $i$ . As a result,

$$\bar{s} = \sqrt[n]{\prod_{i=1}^n \bar{u}_i} \leq \bar{\psi}(0),$$

contradicting the assumption that  $\bar{s} > \bar{\psi}(0)$ .

*Step 2.* Next, fix  $i$  such that  $D_i(0, \bar{u}_i)$  contains at least two elements, which exists due to the argument above. To rule out the second case with  $\bar{s} > \bar{\psi}(0)$ , we construct a perturbation  $\Delta : [0, 1] \rightarrow \mathbf{R}^n$  so that  $\mathbf{x} \pm \varepsilon \Delta$  is a feasible reduced form provided  $\varepsilon > 0$  is small enough. As in the proof of Lemma 5, the existence of such  $\Delta$  contradicts extremality.

To this end, let  $\underline{u}^*, \bar{u}^*$  be the last two points in  $D_i(0, \bar{u}_i)$ , where  $\bar{u}^* = \bar{u}$  as explained above. Denote the value of

$$\max_{\mathbf{u} \in [0, 1]^n} B_x(\mathbf{u}) \quad \text{s.t.} \quad \mathbf{u} \in \prod_{j=1}^n D_j(0, \bar{u}_j), \quad \mathbf{u} \neq \bar{\mathbf{u}}$$

by  $\omega$ . Due to finiteness of bidders' discontinuity sets and  $G < 1$  on  $[0, \bar{s})$ ,  $\omega$  is well-defined and strictly below 1.

First of all, for each  $j \neq i$ , set  $\Delta_j = 0$ . For coordinate  $i$ , set

$$\Delta_i(u) = \kappa \cdot \frac{\mathbf{1}_{[\underline{u}^*, \bar{u}^*)}(u)}{\bar{u}^* - \underline{u}^*}$$



for some  $\kappa > 0$  small enough to ensure  $x_i \pm \Delta_i \in \mathcal{X}$ . For example, the choice analogous to (41) will do.

Next, pick any choose  $\varepsilon \in (0, 1)$  such that  $\varepsilon \leq \bar{\varepsilon}$ , where

$$\bar{\varepsilon} = \frac{1 - \omega}{\max_{u_i \in [\underline{u}^*, \bar{u}^*]} \left| \int_{u_i}^{\bar{u}^*} \Delta_i(u) du \right|} > 0.$$

Since  $\mathbf{x}$  is piecewise constant on  $[0, \bar{\mathbf{u}}]$ , the mapping

$$u_j \in [0, \bar{u}_j] \mapsto B_{\mathbf{x} \pm \varepsilon \Delta}(\mathbf{u})$$

is piecewise linear holding other coordinates  $(u_1, \dots, u_{j-1}, u_{j+1}, \dots, u_n)$  fixed in  $\prod_{k \neq j} [0, \bar{u}_k]$ . As a result, (36) is equivalent to

$$B_{\mathbf{x} \pm \varepsilon \Delta}(\mathbf{u}) \leq 1, \quad \forall \mathbf{u} \in \prod_{j=1}^n D_j(0, \bar{u}_j). \quad (43)$$

By construction, our perturbation  $\Delta$  verifies (43) at all jump points including  $\bar{\mathbf{u}}$  (due to the definition of  $\Delta$ ), and hence (36) holds as well. But then Lemma 3 implies that  $\mathbf{x} \pm \varepsilon \Delta$  is feasible, which contradicts extremality of  $\mathbf{x}$ .

To sum up, we have shown that Case 2 with  $\bar{s} > \bar{\psi}(0)$  is inconsistent with extremality. We now turn to Case 1, where we consider a sag  $(\underline{s}, \bar{s})$  so that  $G$  equals 1 at its endpoints. As before, we denote  $\bar{\mathbf{u}} = \boldsymbol{\nu}(\bar{s})$  and  $\underline{\mathbf{u}} = \boldsymbol{\nu}(\underline{s})$ , which satisfy  $\prod_{i=1}^n \bar{u}_i > 0$  due to  $\bar{s} > 0$  and  $\prod_{i=1}^n \underline{u}_i > 0$  whenever  $\underline{s} > 0$ . Below, we study  $\underline{s} > 0$  and  $\underline{s} = 0$  separately. Lemma 7 does not cover the latter.

*Case 1 with  $\underline{s} > 0$ .* By Lemmas 5, 6, 7, for each  $i$  with  $\underline{u}_i < \bar{u}_i$ , the function  $x_i$  is piecewise-constant on this interval with finitely many jumps including  $\underline{u}_i, \bar{u}_i$ .

By Lemma 8, we can find two distinct coordinates  $i \neq j$  such that  $\underline{u}_i < \bar{u}_i$  and  $\underline{u}_j < \bar{u}_j$ . Following the logic of Step 2, we shall construct a perturbation to contradict that extremality of  $\mathbf{x}$  assumed in the premise.

Let  $\underline{u}_i^*, \bar{u}_i^*$  and  $\underline{u}_j^*, \bar{u}_j^*$  be the last two points  $D_i(\underline{u}_i, \bar{u}_i)$  and  $D_j(\underline{u}_j, \bar{u}_j)$ , respectively, where  $\bar{u}_i^* = \bar{u}_i$  and  $\bar{u}_j^* = \bar{u}_j$ . As before, denote the value of

$$\max_{\mathbf{u} \in [0, 1]^n} B_{\mathbf{x}}(\mathbf{u}) \quad \text{s.t.} \quad \mathbf{u} \in \prod_{k=1}^n D_k(\underline{u}_k, \bar{u}_k), \quad \mathbf{u} \neq \underline{\mathbf{u}}, \bar{\mathbf{u}}.$$

by  $\omega$ , which is again well-defined and strictly less than 1 due to  $G < 1$  on  $(\underline{s}, \bar{s})$ .

Now, define a perturbation  $\Delta : [0, 1] \rightarrow \mathbf{R}^n$  by setting  $\Delta_k = 0$  for all indices  $k$  other than  $i, j$ . For coordinates  $i$  and  $j$ , set

$$\Delta_i(u) = \kappa \cdot \frac{\mathbf{1}_{[\underline{u}_i^*, \bar{u}_i^*)}(u)}{\bar{u}_i^* - \underline{u}_i^*}, \quad \Delta_j(u) = -\kappa \cdot \frac{\mathbf{1}_{[\underline{u}_j^*, \bar{u}_j^*)}(u)}{\bar{u}_j^* - \underline{u}_j^*}$$

for some  $\kappa > 0$  small enough to ensure  $x_i \pm \Delta_i \in \mathcal{X}$  and  $x_j \pm \Delta_j \in \mathcal{X}$ . For example, the minimum of two values defined analogously to (41) for coordinates  $i$  and  $j$  will suffice.

Note that

$$\int_{\underline{u}_i}^{\bar{u}_i} \Delta_i(u) du + \int_{\underline{u}_j}^{\bar{u}_j} \Delta_j(u) du = 0,$$

which implies

$$B_{\mathbf{x} \pm \varepsilon \Delta}(\mathbf{u}) \leq 1 \quad \forall \mathbf{u} \in \prod_{k=1}^n D_k(\underline{u}_k, \bar{u}_k), \quad (44)$$

for all  $\varepsilon \in (0, 1)$  such that  $\varepsilon \leq \bar{\varepsilon}$ , where

$$\bar{\varepsilon} = \frac{1}{2} \frac{1 - \omega}{\max \left\{ \max_{u_i \in [\underline{u}_i^*, \bar{u}_i^*]} \left| \int_{\underline{u}_i}^{\bar{u}_i^*} \Delta_i(u) du \right|, \max_{u_j \in [\underline{u}_j^*, \bar{u}_j^*]} \left| \int_{\underline{u}_j}^{\bar{u}_j^*} \Delta_j(u) du \right| \right\}} > 0.$$

Since each function  $x_k$  is piecewise constant on  $[\underline{u}_k, \bar{u}_k]$ , the argument in Case 2 implies that (44) is equivalent to (39). We conclude that  $\mathbf{x} \pm \varepsilon \mathbf{\Delta}$  is feasible due to Lemma 4, which contradicts extremality of  $\mathbf{x}$ .

*Case 1 with  $\underline{s} = 0$ .* Since  $\underline{s} = 0$ , we have  $G(0) = 1$  which means  $\bar{\psi}(0) = 0$  because  $G < 1$  on  $[0, \bar{\psi}(0))$  due to Lemma 2 and the proof of Theorem 1. Since  $x_i = 0$  on  $[0, \underline{u}_i)$ , the discontinuity points up to  $\bar{u}_i$  are the same whether you start at  $\underline{u}_i$  or 0, exactly as happens in Case 2. By the first step in the proof of this case, the discontinuity set  $D_i(0, \bar{u}_i)$  contains at least two points for some index  $i$ . Furthermore, by the second step, there cannot be two such indices, hence we have

$$D_j(0, \bar{u}_j) = \{\bar{u}_j\} \quad \forall j \neq i.$$

We proceed in two logical steps. First, we argue that  $D_i(0, \bar{u}_i)$  must contain at least three elements, and then we complete the proof by ruling out this possibility as well.

*Step 1.* By the way of contradiction, suppose that  $D_i(0, \bar{u}_i)$  contains exactly two discontinuity points, where one of them is necessarily  $\bar{u}_i$  due to Lemma 6. Let  $\underline{u}_i^* < \bar{u}_i$  be the other discontinuity of  $x_i$ .

Suppose first that  $\underline{u}_i^* > 0$  is strictly positive. Then, for each sufficiently small  $s > 0$ , i.e., any value strictly less than  $\sqrt[n]{\underline{u}_i^* \prod_{j \neq i} \bar{u}_j}$  will do, we have

$$G(s) = s^n + \sum_{j=1}^n \int_0^1 x_j(u) du$$

because  $x_j = 0$  on  $[0, \bar{u}_j)$  for all  $j$  other than  $i$ . But this implies that

$$\sum_{j=1}^n \int_0^1 x_j(u) du < 1$$

contradicting the fact that  $G(\underline{s}) = 1$ , since  $\bar{\psi}(0) = \underline{s} = 0$ .

It follows that we must have  $\underline{u}_i^* = 0$ , meaning that  $x_i$  jumps at  $u = 0$  and  $u = \bar{u}_i$ . Let  $\chi$  denote the size of the first jump, that is the constant value of  $x_i$  on the whole interval  $[0, \bar{u}_i)$ . Recollect that  $G$  equals 1 at the endpoints of the sag  $(\underline{s}, \bar{s})$ , which can be expressed as

$$\bar{u}_i \cdot \chi + \sum_{j=1}^n \int_{\bar{u}_j}^1 x_j(u) du = \bar{u}_i \cdot \prod_{j \neq i} \bar{u}_j + \sum_{j=1}^n \int_{\bar{u}_j}^1 x_j(u) du = 1. \quad (45)$$

One immediate implication of (45) is that the jump size of  $x_i$  satisfies

$$\chi = \prod_{j \neq i} \bar{u}_j.$$

Take any  $s \in (0, \bar{s})$  and consider  $\mathbf{u}$  that equals  $\bar{\mathbf{u}}$  for all coordinates but

$$u_i = \frac{\bar{u}_i}{\bar{s}^n} s^n.$$

Evaluate Border's constraint at this point to obtain

$$B(\mathbf{u}) = s^n + (\bar{u}_i - u_i) \cdot \chi + \sum_{j=1}^n \int_{\bar{u}_j}^1 x_j(u) du = 1,$$

where the last equality follows from the definition of  $\chi$ ,  $u_i$ , and (45). Since  $\mathbf{u}$  is feasible in the problem defining  $G(s)$  in Lemma 2, we must have  $G(s) = 1$ , which contradicts the assumption that  $G < 1$  on  $(\underline{s}, \bar{s})$  as  $\underline{s} = 0$ .

*Step 2.* We have established that there are at least three elements in  $D_i(0, \bar{u}_i)$ . To conclude the proof, we shall again invoke a perturbational argument that has been extensively used earlier in this proof as well as to establish the auxiliary lemmas.

Let  $\underline{u}_i^* < \bar{u}_i^* < \bar{u}_i$  be the last three such points, where the fact that  $x_i$  jumps at the right endpoint is due to Lemma 6. Denote the value in the following program

$$\max_{\mathbf{u} \in [0,1]^n} B_{\mathbf{x}}(\mathbf{u}) \quad \text{s.t.} \quad \mathbf{u} \in \prod_{j=1}^n D_j(0, \bar{u}_j), \quad \mathbf{u} \neq \underline{\mathbf{u}}, \bar{\mathbf{u}}$$

by  $\omega$ , which is well-defined because each discontinuity set is finite, and is strictly less than one because  $G < 1$  on the interior of the sag.

Consider a perturbation  $\Delta : [0, 1] \rightarrow \mathbf{R}^n$  used in the proof of the second part of Lemma 5, that is  $\Delta_j = 0$  for all  $j \neq i$ , whereas for coordinate  $i$ , we have

$$\Delta_i(u) = \kappa \cdot \left( \frac{\mathbf{1}_{[\underline{u}_i^*, \bar{u}_i^*)(u)}}{\bar{u}_i^* - \underline{u}_i^*} - \frac{\mathbf{1}_{[\bar{u}_i^*, \bar{u}_i)(u)}}{\bar{u}_i - \bar{u}_i^*} \right)$$

for some  $\kappa > 0$  small enough to ensure  $\Delta_i \neq 0$  and  $x_i \pm \Delta_i \in \mathcal{X}$ . Again, the choice analogous to (41) will do.

By construction,

$$\int_{\underline{u}_i^*}^{\bar{u}_i} \Delta_i(u) du = 0,$$

Hence the perturbation preserves the relevant integral constraints, and Border's constraint at any grid point in  $\prod_{j=1}^n D_j(0, \bar{u}_j)$ . In particular, we obtain

$$B_{\mathbf{x} \pm \varepsilon \Delta}(\mathbf{u}) \leq 1 \quad \forall \mathbf{u} \in \prod_{j=1}^n D_j(0, \bar{u}_j) \quad (46)$$

for every  $\varepsilon \in (0, 1)$  such that  $\varepsilon \leq \bar{\varepsilon}$ , where

$$\bar{\varepsilon} = \frac{1 - \omega}{\max_{u_i \in [\underline{u}_i^*, \bar{u}_i]} \left| \int_{u_i}^{\bar{u}_i} \Delta_i(u) du \right|} > 0.$$

Similarly to Case 2, since each function  $x_j$  is piecewise constant on  $[0, \bar{u}_j]$ , (46) is equivalent to (36). Thus, by Lemma 3, both  $\mathbf{x} \pm \varepsilon \Delta$  are feasible, contradicting extremality of  $\mathbf{x}$ .

*Putting all pieces together.* To sum up, we have established that if  $\mathbf{x}$  is an extremal reduced form, then we cannot have any sag  $(\underline{s}, \bar{s})$  with  $\bar{s} > \bar{\psi}(0)$ . But this means that

$$G(s) = 1 \quad \forall s \in [\bar{\psi}(0), 1], \quad (47)$$

which as explained in the proof of Lemma 2 is equivalent to Border's constraint being identically 1 along the whole

principal curve.

As explained in Footnote 12, the  $\psi$ -transform can be inverted to obtain the underlying CDF in  $\mathcal{X}$ . It is easy to see that  $\bar{x} = \bar{\psi}^{-1}/u$  is the CDF corresponding to  $\bar{\psi}$ . Using  $\bar{x}$ , we can unpack (47) for any  $s \geq \bar{\psi}(0)$  as

$$\begin{aligned} \frac{1 - s^n}{n} &= \int_{\bar{\psi}^{-1}(s)}^1 \iota d \ln \bar{\psi}(\iota) \\ &= \int_{\bar{\psi}(\bar{\psi}^{-1}(s))}^1 \bar{x}(u) du \\ &= \int_s^1 \bar{x}(u) du, \end{aligned}$$

where the first line uses (9) in Theorem 1, the second one is due to Lemma 1, and the third line follows from strict monotonicity of  $\bar{\psi}^{-1}$  on  $[\bar{\psi}(0), 1]$ . But this means that  $\bar{x}(u)$  equals  $u^{n-1}$  above  $\bar{\psi}(0)$  and is zero below that threshold, which is equivalent to (II) in the theorem.  $\square$

### A.2.3 Proofs of Theorems 2 and 3: Statement II

*Proof.* Take a feasible reduced form  $\mathbf{x}$  such that the geometric average of its  $\psi$ -transforms denoted  $\boldsymbol{\psi} = (\psi_1, \dots, \psi_n)$  satisfies (II). By the definition of the  $\psi$ -transform and its bijectivity (see Footnote 21), we have  $ux_i(u) = \psi_i^{-1}(u)$  for all bidders  $i$ .

Fix an index  $i$  and let  $\tilde{x}_i$  be the interim winning probability of that bidder induced the score allocation described in Theorem 3, that is

$$\tilde{x}_i(u) = \begin{cases} \prod_{j \neq i} \mathbb{E} \left[ \mathbf{1}_{[0, \psi_i^{-1}(u))} \left( \psi_j^{-1}(u_j) \right) \right] & \text{if } u \geq \psi_i(0), \\ 0 & \text{if } u < \psi_i(0). \end{cases}$$

We shall show that  $x_i$  and  $\tilde{x}_i$  coincide. Clearly, both these functions equal 0 on  $[0, \psi_i(0))$ , and so it suffices to consider their values on the complement of that set.

Let  $u \geq \psi_i(0)$ . Since for each  $j$ , by definition,  $\psi_j^{-1}$  is zero on  $[0, \psi_j(0))$  and strictly increasing on its complement, we have

$$\tilde{x}_i(u) = \prod_{j \neq i} \psi_j \left( \psi_i^{-1}(u) \right).$$

Multiply each side by  $u = \psi_i \left( \psi_i^{-1}(u) \right)$  to obtain

$$\begin{aligned} u \tilde{x}_i(u) &= \prod_{j=1}^n \psi_j \left( \psi_i^{-1}(u) \right) \\ &= \left[ \bar{\psi} \left( \psi_i^{-1}(u) \right) \right]^n \\ &= \max \left\{ \left[ \bar{\psi}(0) \right]^n, \psi_i^{-1}(u) \right\} \\ &= ux_i(u). \end{aligned}$$

where the second line follows from the definition of  $\bar{\psi}$ , the third one is due to  $\bar{\psi}(\iota) = \max \{ \bar{\psi}(0), \sqrt[n]{\iota} \}$ . Finally, the last line is due to  $ux_i(u) = \psi_i^{-1}(u)$  and

$$\psi_i^{-1}(u) \geq \psi_i^{-1}(\psi_i(0)) \geq \left[ \bar{\psi}(0) \right]^n.$$

To see the last inequality, note that  $\psi_i(\iota) = \psi_i(0)$  for  $\iota \leq [\bar{\psi}(0)]^n$  due to monotonicity of  $(\psi_1, \dots, \psi_n)$ .  $\square$

#### A.2.4 Proofs of Theorems 2 and 3: Statement III

*Proof.* This directly follows from the argument in Section 6.2 establish that every score allocation induces an extremal reduced form.  $\square$

### A.3 Optimality

#### A.3.1 Proof of Proposition 1

*Proof.* The proposition follows from combining our results on feasibility and extremality of reduced forms with the discussion of the  $\delta$ -transform in Section 7.

First, as explained in Footnote 21, the  $\delta$ -transforms is a bijection between CDFs on the unit interval and absolutely continuous functions on the nonnegative reals with the slope between 0 and 1. In view of this and the construction of  $R_i$  in the main text,

$$\int_0^1 \sum_{i=1}^n H(x_i(u), u) du = \sum_{i=1}^n \int_0^\infty e^{-t} R_i(\delta_i(t), t) dt,$$

where  $\mathbf{x}$  and  $\boldsymbol{\delta}$  are in this bijective relationship.

Second, by Theorem 1 and Lemma 2, a reduced form  $\mathbf{x}$  is feasible if and only if

$$s^n + n \int_{\bar{\psi}^{-1}(s)}^1 \iota d \ln \bar{\psi}(\iota) \leq 1 \quad \forall s \in [\bar{\psi}(0), 1]. \quad (48)$$

By continuity of the  $\psi$ -transform,  $\bar{\psi}$  is continuous, and hence this condition is equivalent to the following:

$$[\bar{\psi}(s)]^n + n \int_{\bar{\psi}^{-1}([\bar{\psi}(s)]^n)}^1 \iota d \ln \bar{\psi}(\iota) = [\bar{\psi}(s)]^n + n \int_s^1 \iota d \ln \bar{\psi}(\iota) \leq 1 \quad \forall s \in [0, 1],$$

where the equality is due to  $d \ln \bar{\psi}(s) = 0$  on intervals of constancy of  $\bar{\psi}^{-1}([\bar{\psi}(s)]^n)$ . Then, changing variables to  $s = e^{-t}$  and using the definition of the  $\delta$ -transform with

$$\sum_{i=1}^n \delta_i(t) = - \sum_{i=1}^n \ln \psi_i(e^{-t}) = - \ln \left( \prod_{i=1}^n \psi_i(e^{-t}) \right) = -n \ln \bar{\psi}(e^{-t})$$

we can finally obtain

$$\mathbf{x} \text{ is feasible} \iff \int_0^t e^{-\tau} \sum_{i=1}^n \delta'_i(\tau) d\tau \leq 1 - e^{-\sum_{i=1}^n \delta_i(t)},$$

where  $\mathbf{x}$  and  $\boldsymbol{\delta}$  are in the bijective relationship.

These two points imply the first part of the proposition, and then the second part follows from (11) that characterizes extremality of reduced forms.  $\square$

#### A.3.2 Proof of Proposition 2

*Proof.* Take two distinct bidders  $i, j$  and an interval  $(\underline{t}, \bar{t})$  such that  $(\delta_i^*)', (\delta_j^*)'$  are essentially bounded away from 0 and 1 on that interval.

Consider a perturbation  $\Delta : [0, \infty) \rightarrow \mathbf{R}^n$  that is identically zero for all  $k$  other  $i, j$ , and  $\Delta_i$  is a smooth function compactly supported in  $(\underline{t}, \bar{t})$  so that  $\Delta(\underline{t}) = \Delta(\bar{t}) = 0$ , whereas  $\Delta_j$  is the minus of that function. Since  $(\delta_i^*)', (\delta_j^*)'$  are essentially bounded,  $\delta^* \pm \varepsilon \Delta$  is feasible in the optimal control problem in Proposition 1 for all sufficiently small  $\varepsilon > 0$ . As a result, if  $\delta^*$  is a solution, then we necessarily must have

$$\int_{\underline{t}}^{\bar{t}} e^{-t} \left( \frac{\partial}{\partial \delta} R_i(\delta_i^*(t), t) - \frac{\partial}{\partial \delta} R_j(\delta_j^*(t), t) \right) \Delta_i(t) dt = 0.$$

Remark that the term in the brackets is a continuous function. Since the whole integral is zero for all smooth  $\Delta_i$  compactly supported in  $(\underline{t}, \bar{t})$ , we conclude that the term in the brackets is identically zero due to the standard application of the Fundamental lemma of the calculus of variations.  $\square$

## A.4 Optimal auctions: the case of extremality

### A.4.1 Proof of Theorem 4

*Proof.* The proof proceeds in two steps. First, we show that  $\delta^*$  defined in the theorem is indeed a solution to the optimal control problem in Proposition 1. Second, we establish this  $\delta$  corresponds to the auction in which the bidder with the highest nonnegative PVV defined in (25) gets the good with certainty.

*Step 1.* Let  $\lambda = -(p^\#)'$  on  $[0, T)$  and  $\lambda = 0$  above that threshold. By construction,  $\lambda$  is continuously differentiable, strictly positive on  $[0, T)$ , and integrable with  $\int_t^\infty \lambda(\tau) d\tau = \max\{p^\#(t), 0\}$ .

For any  $\delta$  that is feasible in the optimal control problem 1, define

$$\mathcal{C}(\delta) = \int_0^\infty \left[ 1 - e^{-\sum_{i=1}^n \delta_i(t)} - \int_0^t e^{-\tau} \sum_{i=1}^n \delta_i'(\tau) d\tau \right] \lambda(t) dt \quad (49)$$

and remark that

$$\mathcal{C}(\delta) \geq 0$$

because the term in the square brackets is nonnegative.

To proceed, we rewrite  $\mathcal{C}(\delta)$  solely in terms of  $\delta$  rather than its derivatives. First, integrating by parts the second term in (49) and using  $\delta(0) = \mathbf{0}$ , we obtain

$$\mathcal{C}(\delta) = \int_0^\infty \left[ 1 - e^{-\sum_{i=1}^n \delta_i(t)} - e^{-t} \sum_{i=1}^n \delta_i(t) - \int_0^t e^{-\tau} \sum_{i=1}^n \delta_i(\tau) d\tau \right] \lambda(t) dt. \quad (50)$$

Second, changing the order of integration in the third term and using  $\int_t^\infty \lambda(\tau) d\tau = \max\{p^\#(t), 0\}$ , we can further rewrite (50) as

$$\mathcal{C}(\delta) = \int_0^\infty \left[ \left( 1 - e^{-\sum_{i=1}^n \delta_i(t)} - e^{-t} \sum_{i=1}^n \delta_i(t) \right) \lambda(t) - e^{-t} \sum_{i=1}^n \delta_i(t) \max\{p^\#(t), 0\} \right] dt. \quad (51)$$

By construction, the expression (51) is nonnegative for all feasible policies irrespective of their extremality of

optimality, therefore

$$\begin{aligned} \int_0^\infty e^{-t} \sum_{i=1}^n R_i(\delta_i(t), t) dt &\leq \int_0^\infty e^{-t} \sum_{i=1}^n R_i(\delta_i(t), t) dt + \mathcal{C}(\boldsymbol{\delta}) = \int_0^\infty \lambda(t) dt \\ &+ \int_0^\infty e^{-t} \left[ \sum_{i=1}^n R_i(\delta_i(t), t) - \left( e^{t - \sum_{i=1}^n \delta_i^*(t)} + \sum_{i=1}^n \delta_i(t) \right) \lambda(t) - \sum_{i=1}^n \delta_i \max\{p^\sharp(t), 0\} \right] dt. \end{aligned} \quad (52)$$

It is easy to see that  $\mathcal{C}(\boldsymbol{\delta}^*) = 0$ , therefore the bound in (52) is actually tight for this tentative optimal policy. We now make it more permissive while keeping its tightness when evaluated at  $\boldsymbol{\delta}^*$ . Specifically, since the second line (52) is strictly concave in  $\boldsymbol{\delta}$ , we can combine the gradient inequality saying that this concave function lies below its tangent at  $\boldsymbol{\delta}^*$  and  $(-e^{t - \sum_{i=1}^n \delta_i^*(t)} + 1) \lambda(t) = 0$ , which is due to our construction of  $\boldsymbol{\delta}^*$  and  $\lambda$ , to obtain

$$\begin{aligned} \int_0^\infty e^{-t} \sum_{i=1}^n R_i(\delta_i(t), t) dt + \mathcal{C}(\boldsymbol{\delta}) &\leq \int_0^\infty e^{-t} \sum_{i=1}^n R_i(\delta_i^*(t), t) dt \\ &+ \int_0^\infty e^{-t} \sum_{i=1}^n \left[ \underbrace{\frac{\partial}{\partial \delta} R_i(\delta_i^*(t), t) - \max\{p^\sharp(t), 0\}}_{=\omega_i(t)} \right] (\delta_i(t) - \delta_i^*(t)) dt. \end{aligned} \quad (53)$$

We claim that the second line in (53) is maximized at  $\boldsymbol{\delta}^*$  amongst all feasible policies. To see why, fix an index  $i$ , and remark that the term in the square brackets (denoted  $\omega_i$  for short) is identically zero on  $[0, T)$  due to the definition of  $\boldsymbol{\delta}^*$ . Since  $p^\sharp$  is negative on the complement of that set, we have

$$\begin{aligned} \int_0^\infty e^{-t} \omega_i(t) \delta_i(t) dt &= \int_T^\infty e^{-t} \frac{\partial}{\partial \delta} R_i(\delta_i^*(t), t) \delta_i(t) dt \\ &= \left[ \int_T^\infty e^{-t} \frac{\partial}{\partial \delta} R_i(\delta_i^\dagger(t), t) dt \right] \delta_i(T) + \int_T^\infty \left[ \int_t^\infty e^{-\tau} \frac{\partial}{\partial \delta} R_i(\delta_i^\dagger(\tau), \tau) d\tau \right] \delta_i'(t) dt \\ &\leq \left[ \int_T^\infty e^{-t} \frac{\partial}{\partial \delta} R_i(\delta_i^\dagger(t), t) dt \right] \delta_i(T) + \int_T^\infty \left[ \int_t^\infty e^{-\tau} \frac{\partial}{\partial \delta} R_i(\delta_i^\dagger(\tau), \tau) d\tau \right] \delta_i'(t) dt = 0, \end{aligned}$$

where the second line follows from the integration by parts and  $\delta_i^*(t) = \delta_i^*(T) = \delta_i^\dagger(T)$  on  $[T, \infty)$ , whereas the third line is due to concavity of  $R_i$ , the fact that  $\delta_i^\dagger$  is decreasing, and both  $\delta_i(T)$  and  $\delta_i'(t)$  are nonnegative. Clearly, by definition of  $\delta_i^\dagger$ , which shows that the upper bound in (53) is maximized at  $\boldsymbol{\delta}^*$ , thereby establishing optimality of this policy in the optimal control problem.

**Step 2.** Having established optimality of  $\boldsymbol{\delta}^*$ , we now turn to the second part of the theorem. Let  $\boldsymbol{x}^\sharp$  be the reduced form that corresponds to  $\boldsymbol{\delta}^\sharp$ , i.e., by (16), we have

$$u x_i^\sharp(u) = e^{-(\delta_i^\sharp)^{-1}(-\ln u)}. \quad (54)$$

Similarly, let  $\boldsymbol{x}^*$  be the reduced form that corresponds to  $\boldsymbol{\delta}^*$ —it coincides with  $\boldsymbol{x}^\sharp$  for quantiles above the threshold of  $e^{-\delta_i^\sharp(T)}$  and equals zero otherwise.

By construction,  $\boldsymbol{\delta}^\sharp$  satisfies (19) as equality at all times, meaning that its corresponding reduced form is extremal. In view of Theorem 3, it can be induced by awarding the good to bidder  $i$  whenever

$$u_i x_i^\sharp(u_i) > u_j x_j^\sharp(u_j) \quad \forall j \neq i,$$

Equivalently, using (54) and applying  $p^\sharp$ , which is strictly decreasing, to each side, we can rewrite this condition in

terms of bidder's PVVs  $q_i(u_i) = p^\#((\delta_i^\#)^{-1}(-\ln u_i))$  as follows:

$$q_i(u_i) > q_j(u_j) \quad \forall j \neq i.$$

By construction,  $p^\#(T) = 0$ , meaning that  $u_i$  is higher than the threshold of  $e^{-\delta_i^\#(T)}$  if and only if bidder  $i$ 's PVV is nonnegative. This shows that the optimal reduced form  $\mathbf{x}^*$  can be induced by allocating the good to the bidder with the highest PVV provided that it is nonnegative.  $\square$

#### A.4.2 Proof of Proposition 3

*Proof.* This proposition is a special case of Theorem 4 since all conditions of this theorem are verified. Indeed, as explained in the main text, in Myerson's linear model bidders' revenue functions satisfy

$$\frac{\partial}{\partial \delta} R_i(\delta_i, t) = \zeta_i(e^{-\delta_i}),$$

where  $\zeta_i$  is bidder  $i$ 's MVV in the quantile space. Clearly, the environment is regular when bidders' MVV are strictly increasing. Furthermore, the added condition on  $\delta^\dagger$  is satisfied vacuously as

$$\frac{\partial}{\partial \delta} R_i(\delta_i^\dagger(t), t) = 0$$

yields a constant function:  $\delta_i^\dagger(t) = \delta_i^\#(T)$  defined on  $[T, \infty)$ .  $\square$

#### A.4.3 Proof of Proposition 4

*Proof.* Let  $\delta^M$  and  $\delta^*$  be Myerson's solution and the optimum. Clearly, since PVVs are MVVs times the derivative of  $h$ , the cutoff time  $T$  that appears in Theorem 4 is identical for both cases. As explained in the main text, Myerson's solution satisfies

$$\delta_1^M(t) \geq \delta_i^M(t) \quad \forall i \neq 1$$

at all times  $t$ .

Suppose towards a contradiction that there exists some time  $t$  so that  $\delta_1^M(t) > \delta_1^*(t)$ . It is without loss of generality to assume that  $t$  is below the cutoff  $T$ , since both  $\delta^M$  and  $\delta^*$  are frozen after  $T$ . But then there must exist a bidder  $i$  for whom the inequality is reversed at that time, that is  $\delta_i^M(t) < \delta_i^*(t)$ , because  $\sum_{i=1}^n \delta_i^M(t) = \sum_{i=1}^n \delta_i^*(t) = t$ . However, this contradicts (28) and (29) in the main text, and we conclude that  $\delta_1^M(t) \leq \delta_1^*(t)$  for all  $t$ .

Finally, note that  $\delta_1^*$  being larger than  $\delta_1^M$  implies that bidder's interim winning probability at the optimum  $x_1^*$  is larger than in Myerson's auction due to (16). The argument for the weakest bidder is identical and hence omitted.  $\square$

### A.5 Optimal auctions: going beyond extremality

#### A.5.1 Proof of Theorem 5

*Proof.* The proof proceeds is almost identical to the proof of Theorem 4, and we therefore focus on highlighting on key differences instead of repeating the whole argument.



*Step 1.* This steps mimics the first part of the proof of Theorem 4 until (53). Now, under the conditions of this theorem,  $\omega_i$  is identically zero, which directly implies that  $\delta^*$  is a solution to the optimal control problem in Proposition 1.

*Step 2.* As in the proof of Theorem 4, the reduced form  $x^\#$  corresponding to  $\delta^\#$  is extremal; furthermore, it can be implemented by awarding the good to the bidder with the highest PVV. However, the reduced form  $x^*$  corresponding to the optimum  $\delta^*$  is no longer a simple truncation of  $x^\#$ .

By (16), we can express it as

$$ux_i^*(u) = \begin{cases} ux_i^\#(u) = e^{-(\delta_i^\#)^{-1}(-\ln u)} & \text{if } u \geq e^{-\delta_i^\#(T)}, \\ ux_i^\dagger(u) = e^{-(\delta_i^\dagger)^{-1}(-\ln u)} & \text{otherwise.} \end{cases}$$

Our assumptions on  $\delta_i^\dagger$  implies that the function  $x_i^*$  a CDF. Furthermore, we have  $\delta_i^\dagger(t) \leq \delta_i^\#(t)$  on  $[T, \infty)$  due to

$$\frac{\partial}{\partial \delta} R_i(\delta_i^\dagger(t), t) = 0 \geq \frac{\partial}{\partial \delta} R_i(\delta_i^\#(t), t).$$

and strict concavity of  $R_i$ . As a result,  $x_i^\dagger \leq x_i^\#$  on the bottom interval, meaning that the fraction in (31) is well-defined under given our convention in Footnote 28, i.e.,  $r_i(u_i) = 0$  whenever  $x_i^\#(u_i) = 0$ . To sum up, the optimal reduced form  $x^*$  can be induced by first picking a winner according to PVVs—leading to bidders' interim winning probabilities  $x^\#$ —and then allocating the fraction defined in that equation.  $\square$

### A.5.2 Proof of Proposition 5

*Proof.* As explained in the text, it is without loss of generality to look at group symmetric policies due to concavity of bidders' revenue functions, i.e.,  $\delta_1^\#, \delta_n^\#$  and  $\delta_1^\dagger, \delta_n^\dagger$  corresponding to the risk-neutral and risk-averse bidders and defined as described in Theorem 5. Concavity of  $H_i$  in  $x$  and its supermodularity imply that  $\delta_i^\dagger$  is weakly increasing; as a result,  $\delta_i^*$  coincides with  $\delta_i^\#$  before  $T$  and follows  $\delta_i^\dagger$  after that threshold.

We prove the proposition in two steps. First, we shall show that at  $t^o = -\frac{n}{n-1} \ln m < T$ , both  $\delta_1^\#, \delta_n^\# \leq t + \ln m$  if and only if  $t \geq t^o$ . Then, we will rank  $\delta_1^\#, \delta_n^\#$  (as well as  $\delta_1^*, \delta_n^*$ ) before  $t^o$  and after that time, and use this ranking to complete the proof.

**Step 1.** Note that both  $\delta_1^\#, \delta_n^\#$  must be strictly above  $t + \ln m$  for  $t$  sufficiently close to 0 as  $t + \ln m \rightarrow \ln m < 0$  as  $t \rightarrow 0$ . On the other hand, at least one of must be strictly below  $t + \ln m$  for  $t$  sufficiently large as

$$\min\{\delta_1^\#, \delta_n^\#\} \leq \frac{k}{n} \delta_1^\# + \left(1 - \frac{k}{n}\right) \delta_n^\# = \frac{t}{n}.$$

By continuity, the minimum of  $\delta_1^\#, \delta_n^\#$  crosses the line  $t + \ln m$  at some time, say  $t^o$ .

Examination of (32) reveals that at  $t^o$ , we have

$$p^\#(t) = v\left(e^{-\delta_1^\#(t^o)}\right) - v'\left(e^{-\delta_1^\#(t^o)}\right) \left(1 - e^{-\delta_1^\#(t^o)}\right) = v\left(e^{-\delta_n^\#(t^o)}\right) - v'\left(e^{-\delta_n^\#(t^o)}\right) \left(1 - e^{-\delta_n^\#(t^o)}\right),$$

which implies that  $\delta_1^\#(t^o) = \delta_n^\#(t^o) = \frac{t^o}{n} = t^o + \ln m$  due to regularity, (33), and the fact that we have a crossing at  $t^o$ . Thus, at  $t = t^o$ , all  $\delta_i^\#$  are equal to each other and cross the line  $t + \ln m$ . Clearly, this solves to the unique crossing time  $t^o = -\frac{n}{n-1} \ln m$ . This crossing time is below  $T$  since

$$p^\#(t) = v\left(m^{\frac{1}{n-1}}\right) - v'\left(m^{\frac{1}{n-1}}\right) \left(1 - m^{\frac{1}{n-1}}\right) = \frac{\partial}{\partial x} H_i(m, m^{\frac{1}{n-1}}) > 0$$

by assumption.

**Step 2.** Let  $t < t^o$ . Using marginal revenue-equalization and the unimodality assumption on  $g$ , we obtain

$$\frac{\partial}{\partial \delta} R(\delta_1^\#(t), t) = \frac{\partial}{\partial \delta} R_n(\delta_n^\#(t), t) < \frac{\partial}{\partial \delta} R_1(\delta_n^\#(t), t),$$

which implies  $\delta_1^\#(t) > \delta_n^\#(t)$  due to regularity. The symmetric argument applies can be used to show that this ranking of  $\delta_1^\#$ ,  $\delta_n^\#$  is reversed on  $(t, T)$ , and it can also be used to establish that  $\delta_1^\dagger < \delta_n^\dagger$  holds on  $[T, \infty)$

Putting all these pieces together, we conclude:  $\delta_1^*(t) > \delta_n^*(t)$  for  $t < t_0$ ,  $\delta_1^*(t) = \delta_n^*(t)$  for  $t = t_0$  and  $\delta_1^*(t) < \delta_n^*(t)$  for  $t > t_0$ . Finally, using (16), we can obtain  $x_1^*(u) > x_n^*(u)$  for  $u > e^{-\delta_1^*(t_0)} = m^{\frac{1}{n-1}}$ ,  $x_1^*(u) < x_n^*(u)$  for  $u < m^{\frac{1}{n-1}}$  and  $x_1^*(u) = x_n^*(u) (= m)$  for  $u = m^{\frac{1}{n-1}}$ , as claimed.  $\square$

### A.5.3 Example 3

We provide a derivation for the case with 2 risk-neutral bidders and one risk-averse bidder ( $k = 2$ ). For the case  $k = 1$  the derivation is analogous and is omitted.

By marginal revenue equalization in (32), we have

$$2e^{-\delta_1^\#} - 1 = e^{-\delta_3^\#} - (1 - e^{-\delta_3^\#})2e^{\delta_3^\# - t} \quad (55)$$

As  $2\delta_1^\# + \delta_3^\# = t$ , we have  $\delta_1^\# = (t - \delta_3^\#)/2$ . Plugging this in (55), we obtain

$$e^{-\delta_3^\#} - (1 - e^{-\delta_3^\#})2e^{\delta_3^\# - t} = 2e^{(\delta_3^\# - t)/2} - 1. \quad (56)$$

This is equivalent to a fourth-degree polynomial equation in  $e^{-\delta_3^\#/2}$ . However, instead of solving for  $\delta_3^\#$ , we can solve directly for  $x_3^\#$  which is the object we are ultimately interested in. Indeed,  $e^{-\delta_3^\#(t)} = \psi_1(e^{-t})$ , so  $e^{-t} = \psi_1^{-1}(e^{-\delta_3^\#(t)})$ . But notice that  $\psi_i^{-1}(u) = ux_i(u)$ . Thus, solving (56) for  $e^{-t}$  after writing  $e^{-\delta_3^\#} = u$ , we will recover  $ux_3^\#(u)$ . Thus,  $x_3^*(u)$  is the solution to

$$u - (1 - u)2ux_3^\#(u) = 2\sqrt{ux_3^\#(u)} - 1 \Leftrightarrow 2x_3^\#(1 - u) + 2\sqrt{x_3^\#} - u - 1 = 0, \quad (57)$$

so

$$x_3^\#(u) = \frac{2 - u^2 - \sqrt{3 - 2u^2}}{2(1 - u)^2}. \quad (58)$$

By Theorem 5, the expression in (58) will be optimal only when  $\delta_3^*(t) = \delta_3^\#(t)$ , that is,  $t \leq T$ . To find  $T$ , we equate both sides of (56) to 0 which yields  $\delta_1^\#(T) = \ln 2$ ,  $\delta_3^\#(T) - T = -2 \ln 2$ , so  $e^{-\delta_3^\#(T)} - (1 - e^{-\delta_3^\#(T)})2(1/4) = 0$ ,  $\delta_3^\#(T) = \ln 3$ . Thus,  $T = 2\delta_1^\#(T) + \delta_3^\#(T) = \ln 12$ . The expression in (58) is valid for  $u \geq e^{-\delta_3^\#(T)} = 1/3$ .

To find  $x_1^\#(u)$ , we can do the same by writing (55) in terms of  $\delta_1$  only. We get

$$2e^{-\delta_1^\#} - 1 = e^{2\delta_1^\# - t} - (1 - e^{2\delta_1^\# - t})2e^{-2\delta_1^\#}.$$

Writing  $e^{-\delta_1^\#} = u$  and solving for  $e^{-t} = ux_1^\#$ , we get

$$x_1^\#(u) = \max \left\{ \frac{2u + 2u^2 - 1}{2u + 1/u}, 0 \right\}.$$

This is optimal for  $u \geq e^{-\delta_1^\#(T)} = 1/2$ .

For lower  $u$ , the optimal allocations are governed by  $\delta_i^\dagger(t)$  which can be found from  $\frac{\partial}{\partial \delta} R_i(\delta_i^\dagger(t), t) = 0$ , which boils down to just  $\frac{\partial}{\partial x} H_i(x_i^\dagger(u), u) = 0$ . Thus, for  $u < 1/3$   $u - (1 - u)2x_3^\dagger(u) = 0$ , so  $x_3^\dagger = \frac{u}{2(1-u)}$ . Thus, by Theorem 5,

$$x_3^*(u) = \begin{cases} \frac{u}{2(1-u)}, & u < 1/3; \\ \frac{2-u^2-\sqrt{3-2u^2}}{2(1-u)^2}, & u \geq 1/3. \end{cases}$$

$x_3^*(u)$  is continuous. (To apply Theorem 5, we have to check that the derivative of  $\delta_3^\dagger(t)$  is (weakly) between zero and one, but here we do not have to solve for  $\delta_3^\dagger(t)$  explicitly to do that. Indeed, as the resulting  $x_3^\dagger(u) = \frac{u}{2(1-u)}$  is increasing, we know by the general properties of delta transforms that its delta transform satisfies  $\delta_3'(t) \in [0, 1]$ .)

Likewise, for  $u < 1/2$ , we find  $\delta_1^\dagger(t) = \ln 2$  for  $t \geq T$ , which clearly has a derivative between 0 and 1. Recall that constant delta transforms are generated by jumps in the allocation  $x$ . In fact, by (16),  $x_i^\dagger(u) = \exp\left(-(\delta_i^\dagger)^{-1}(-\ln u)\right)/u$  we get that for  $u < 1/2$ ,  $(\delta_1^\dagger)^{-1}(-\ln u) = +\infty$ , as  $-\ln u > \ln 2 = \delta_1^\dagger(t)$  for all  $t$ . Thus,  $x_1^\dagger(u) = \exp(-\infty)/u = 0$  for  $u < 1/2$ .

Summing up, the optimal allocation to a risk-neutral bidder is

$$x_1^*(u) = x_2^*(u) = \begin{cases} 0, & u < 1/2; \\ \frac{2u+2u^2-1}{2u+1/u}, & u \geq 1/2. \end{cases}$$

At  $u = 1/2$ , the interim allocation experiences a jump from 0 to  $1/6$ .

## A.6 Geometry of the $\delta$ -transform

The  $\delta$ -transform yields an explicit geometric decomposition of a CDF  $x \in \mathcal{X}$  into: (a) vertical segments (jumps), corresponding to  $\delta' = 0$ ; (b) horizontal segments with no mass, corresponding to  $\delta' = 1$ ; (c) strictly increasing continuous segments, corresponding to  $\delta' \in (0, 1)$ . For example, Figure 1 plots the  $\psi$ -transform corresponding to

$$x(u) = u \cdot \mathbf{1}_{[1/4, 1/2) \cup [3/4, 1)}(u) + 1/2 \cdot \mathbf{1}_{[1/2, 3/4)}(u),$$

which (a) jumps at  $u = 1/4$  and  $u = 3/4$ , (b) is horizontal on  $[0, 1/4)$  and  $[1/2, 3/4)$ , and (c) coincides with the diagonal on the remaining intervals. The resulting  $\delta$ -transform satisfies

$$\delta'(t) = \begin{cases} 0, & t \in [2 \ln(4/3), \ln(8/3)) \cup [2 \ln 4, \infty), \\ 1, & t \in [\ln(8/3), 2 \ln 2), \\ 1/2, & t \in [0, 2 \ln(4/3)) \cup [2 \ln 2, 2 \ln 4), \end{cases}$$

and is illustrated in Figure 3 below. Because the  $\delta$ -transform proceeds from  $u = 1$  downward, the first flat interval of  $x$  appears as the last interval where  $\delta' = 0$ , and similarly for the other pieces of the decomposition.

## A.7 Regularity

In this section, we unpack regularity in terms of curvature properties of  $H_i$ .

**Condition (A).** We measure how  $\frac{\partial H_i}{\partial x}$  responds to changes in its arguments through the following pair of semi-elasticities:

$$\xi_i^x = \frac{\partial}{\partial \ln x} \frac{\partial H_i}{\partial x}, \quad \xi_i^u = \frac{\partial}{\partial \ln u} \frac{\partial H_i}{\partial x}.$$

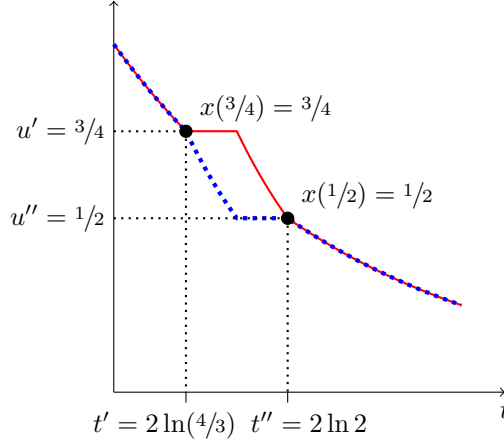


Figure 3: Reading of the CDF  $x(u)$  in Figure 1 for  $u = u', u''$  from the parametric curves  $e^{-\delta(t)}$  and  $e^{t-\delta(t)}$  shown in red (solid) and blue (dotted).

By definition of bidders' revenue functions  $R_i$  in (18), we have  $\frac{\partial}{\partial \delta} R_i(\delta, t) = \frac{\partial}{\partial x} H_i(e^{\delta-t}, e^{-\delta})$ , hence

$$\frac{\partial^2}{\partial \delta^2} R_i(\delta, t) = \xi_i^x(e^{\delta-t}, e^{-\delta}) - \xi_i^u(e^{\delta-t}, e^{-\delta}), \quad \frac{\partial^2}{\partial \delta \partial t} R_i(\delta, t) = -\xi_i^x(e^{\delta-t}, e^{-\delta}). \quad (59)$$

In view of these identities, bidder  $i$ 's revenue function  $R_i$  is strictly concave if the  $x$ -semi-elasticity is strictly smaller than the  $u$ -semi-elasticity. This holds vacuously when  $H_i$  is strictly supermodular and concave, since then  $\xi_i^x \leq 0 < \xi_i^u$ . More generally, Condition (A) requires that supermodularity is sufficiently strong relative to convexity.

**Condition (B).** We now assume Condition (A) and examine the added restrictions imposed by Condition (B). Under strict concavity of  $R_i$ , one can uniquely solve (23) for functions  $(\delta^\#, p^\#)$ . Then, regularity requires  $\delta^\#$  to be strictly increasing and  $p^\#$  to be strictly decreasing. To see when it is the case, differentiate (23) using the Implicit function theorem to obtain

$$\frac{\partial^2 R_i^\#}{\partial \delta^2} \cdot (\delta_i^\#)' + \frac{\partial^2 R_i^\#}{\partial \delta \partial t} = (p^\#)'(t), \quad \sum_{i=1}^n (\delta_i^\#)' = 1, \quad (60)$$

where  $R_i^\#$  stays for  $R_i$  evaluated at  $\delta_i^\#(t)$  for each time  $t$ . Let  $\xi_j^{x,\#}$  and  $\xi_j^{u,\#}$  be the semi-elasticities evaluated along the marginal revenue equalizing path, and set

$$\kappa^\# = \sum_{j=1}^n \frac{1}{\xi_j^{x,\#} - \xi_j^{u,\#}}.$$

Using (59), the above linear system solves to

$$\underbrace{\kappa^\# \cdot (\xi_i^{x,\#} - \xi_i^{u,\#})}_{>0} \cdot (\delta_i^\#)' = 1 - \sum_{j=1}^n \frac{\xi_j^{x,\#} - \xi_i^{x,\#}}{\xi_j^{x,\#} - \xi_j^{u,\#}}, \quad \underbrace{\kappa^\#}_{<0} \cdot (p^\#)' = 1 - \sum_{j=1}^n \frac{\xi_j^{x,\#}}{\xi_j^{x,\#} - \xi_j^{u,\#}}, \quad (61)$$

meaning that the strict positivity of right-hand sides in (61) is necessary and sufficient for Condition (B) provided that Condition (A) is satisfied.

To sum up, the added Condition (B) is verified if and only if

$$1 > \sum_{j=1}^n \frac{\xi_j^{x,\#} - \max\{0, \xi_1^{x,\#}, \dots, \xi_n^{x,\#}\}}{\xi_j^{x,\#} - \xi_j^{u,\#}} \quad (62)$$

along the path equalizing bidders' marginal revenues. This trivially holds in Myerson's linear model as  $x$ -semi-elasticity is identically zero. More generally, (62) demands that the semi-elasticities cannot be too strong and differ across various bidders too much along  $\delta^\#$ .

As we pointed out earlier, Condition (A) is satisfied vacuously when  $H_i$  strictly supermodular and concave in its first argument. Then, since the  $x$ -semi-elasticity is negative, (62) simplifies to

$$1 > \sum_{j=1}^n \frac{\xi_j^{x,\#}}{\xi_j^{x,\#} - \xi_j^{u,\#}}.$$

If, in addition, the environment is symmetric, then the marginal revenue equalizing path is necessarily symmetric, i.e.,  $\delta_i^\# = t/n$ , which gives  $x_i^\# = u^{n-1}$ . We can further rewrite it as

$$1 > n \frac{\xi^x(u^{n-1}, u)}{\xi^x(u^{n-1}, u) - \xi^u(u^{n-1}, u)},$$

which nests the regularity condition in Theorem 1 of [Gershkov et al. \[2022\]](#) for the symmetric CRA specification:  $H(x, u) = v(u)x - (1 - u)v'(u)g(x)$ .