The APS Approach for Undiscounted Quitting Games*

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Abstract

Characterizing and explicitly computing equilibria of undiscounted dynamic games have been a challenge for many years. In this paper, we study *quitting games*, which are stopping games where the terminal payoff does not depend on the stage of termination. We adapt the recursive approach of Abreu, Pearce, and Staccheti (1990) to characterize a certain subset of the set of subgame-perfect ε-equilibrium payoffs. Our approach is based on the novel representation of strategy profiles through *absorption paths*, which was developed in Ashkenazi-Golan, Krasikov, Rainer, and Solan (2023), and our characterization focuses on absorption paths in which exactly one player randomizes between quitting and continuing at any point in time. We then adapt the results to larger classes of absorption paths. Since quitting games form a special case of both stopping games and stochastic games, our approach may be useful in studying more general classes of these games.

Keywords: Stochastic games, quitting games, continuous equilibria, perfect monitoring, monotone convex set-valued operators.
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1 Introduction

Abreu, Pearce, and Staccheti (1990) developed the so-called *APS approach*, which allows characterizing the set of equilibria in discounted games using the largest fixed point of the

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Shapley operator.[1] Their ideas have found numerous applications in economics and game theory.[2] Unfortunately, the APS approach is invalid when the game is undiscounted. In this paper, we make, to our knowledge, the first attempt to adapt the APS approach to undiscounted games. We do that for an important class of stochastic games called *quitting games*.

Quitting games are stochastic games with a single non-absorbing state,where at every stage each player decides whether to continue the interaction or quit. Once at least one player quits, the game moves to an absorbing state. Thus, quitting games are also a simple family of stopping games.

The significance of quitting games stems from two reasons. First, since they form a class of both stochastic games and stopping games, results proven for quitting games can be extended to more general families of stopping games and stochastic games; compare, for example, Solan and Vohra (2001) to Solan and Vohra (2002), or Solan and Solan (2020) to Solan and Solan (2021), or even Simon (2012) to Simon (2007) (which was developed after Simon (2012)). Second, since quitting games are a simple class of games, they are easier to study than general stochastic games and general stopping games. Yet, they are sufficiently complex; so many dynamic aspects of general stopping and stochastic games remain, making their analysis nontrivial. Quitting games proved in the past to be fertile soil for using new mathematical methods, e.g., dynamical systems (Solan and Vieille, 2001), differential inclusions (Solan, 2005), topological tools (Simon, 2012), linear complementarity problems (Solan and Solan, 2020), or the concept of absorption paths (Ashkenazi-Golan, Krasikov, Rainer, and Solan, 2023).

When players discount their payoffs, both stochastic games and stopping games admit equilibria under fairly general conditions (see, e.g., Fink (1964), Takahashi (1964), Mertens and Parthasarathy (1987), Ferenstein (2007), Jaskiewicz and Nowak (2018)). The seminal papers of Fink (1964) and Takahashi (1964) characterize the set of stationary equilibria of discounted stochastic games with finite or countable infinite state space and finite action space as fixed points of the Shapley operator. Abreu, Pearce, and Stacchetti (1990) further extended this approach to all discounted equilibria of repeated games with imperfect monitoring. In contrast, when players do not discounted their payoffs, even the existence of equilibria remains an open question that has been answered positively only under restrictive conditions (for stochastic games, see, e.g., Vrieze and Thuijsman (1989), Solan (1999), Vieille (2000a, 2000b), Simon (2007, 2012), and Flesch, Schoenmakers, and Vrieze (2008, 2009), and for stopping games, see, e.g., Shmaya and Solan (2004)).

The approach of Abreu, Pearce, and Stacchetti (1990) has been successfully used by Hörner, Sugaya, Takahashi, and Vieille (2011) and Fudenberg and Yamamoto (2011) to prove a Folk theorem for discounted stochastic games with imperfect public monitoring of states and actions. Both papers assumed certain ergodicity property on the transitions. This condition is not satisfied if a stochastic game has absorbing states, and quitting games are arguably the simplest example of such games.

[1] The original paper dealt with repeated games. However, their approach mechanically extends to discounted stochastic games with a state variable.

[2] Two notable examples are risk-sharing (Kocherlakota (1996)) and unemployment insurance (Thomas and Worrall (2007).

In addition, the APS approach has triggered the development of numerical methods that quantitatively characterize discounted equilibria in stochastic games. There are several efficient algorithms for computing equilibria of discounted stochastic games with public randomization, e.g., Yeltekin, Cai, and Judd (2017) and Abreu, Brooks, and Sannikov (2020), but methods for games without public randomization are much more sparse. The notable exceptions are Berg (2019) and Berg and Kitti (2019) who studied repeated games without any state variables. In the present paper, we do not allow for public correlation and study undiscounted quitting games in which players' payoffs depend on a nontrivial state variable, i.e., the set of players who quit at the termination stage.

In their seminal work, Flesch, Thuijsman, and Vrieze (1997) studied a specific three-player quitting game and found that all undiscounted equilibria of the game have a cyclic structure. This phenomenon was further extended by Solan (1999) to all three-player absorbing games, where it was found that in this class of games, there is always an undiscounted equilibrium where the equilibrium play is either stationary or cyclic. Simon (2007) observed that when more than three players participate in the quitting game, the equilibria consist of two different behaviors. There are stages where some players quit with probability bounded away from 0. In such stages, the players play an equilibrium in the one-shot game, where the payoff if no one quits is the continuation payoff. In the remaining stages, the players quit with low probability, as if the game is played in continuous time.

To accommodate these two kinds of behavior, Ashkenazi-Golan, Krasikov, Rainer, and Solan (2023; AKRS for short) introduced the concept of absorption paths, which is an alternative representation of strategy profiles that allows for both discrete-time aspects and continuous-time aspects in the player behavior, and involves re-parameterizing time according to the accumulated probability of absorption.

AKRS introduced a notion of subgame perfectness that applies to absorption paths, which they called sequential perfectness to distinguish it from subgame perfectness that applies to strategy profiles, and showed that the set of payoffs that correspond to sequentially perfect absorption paths coincides with the set of subgame-perfect equilibrium payoffs, namely, limits of subgame-perfect ε-equilibrium payoffs as ε goes to 0.

In the present paper, using the Shapley operator, we characterize the set of payoffs, which correspond to sequentially perfect absorption paths that involve only continuous-time aspects. This set corresponds to the limit set of subgame-perfect ε-equilibrium payoffs, where along the corresponding subgame-perfect ε-equilibria, players quit with infinitesimal probabilities. We note that such equilibria need not exist; see, e.g., Solan and Vieille (2002).

The paper is organized as follows. The model is described in Section 2. A naive approach to characterizing the set of undiscounted equilibrium payoffs appears in Section 3. The more sophisticated approach using the *Essential APS* operator, is described in Section 4. Extensions of the approach to equilibria in which a set of players is allowed to randomize quitting in continuous time are presented in Section 5. Discussions on possible extensions of our results to all equilibrium payoffs, as well as conclusions and final remarks, appear in Section 6.

2 Model

This section presents the main ingredients of the model. Quitting games are defined in Section 2.1. In Section 2.2 we describe the APS approach and explain why its naive application to the undiscounted game fails. This leads to a new concept of strategy profiles, called Flesch absorption paths (FAPs), which is motivated in Section 2.3 and is formally defined in Section 2.4.

2.1 Quitting Games

In this subsection, we recall the setting of quitting games as it was introduced in Solan and Vieille (2001).

Definition 2.1 *A quitting game is a pair $\Gamma = (I, r)$, where I is a finite set of players with $|I| \geq 2$ and $r : \prod_{i \in I} \{C_i, Q_i\} \to \mathbb{R}^I$ is a payoff function.*

Player i's action set is $A_i := \{C_i, Q_i\}$. These actions are interpreted as continue and quit, respectively. Denote by $A := \prod_{i \in I} A_i$ the set of action profiles. The game is played as follows. Let $\mathbb{N} = \{0, 1, 2, \ldots\}$ be the set of nonnegative integers. At every stage $n \in \mathbb{N}$ each player $i \in I$ chooses an action $a_i^n \in A_i$. If all players continue, the game continues to the next stage; if at least one of the players quits, the game terminates, and the terminal payoff is $r(a^n)$, where $a^n = (a_i^n)_{i \in I}$. If no player ever quits, the payoff is $r(\vec{C})$, where $\vec{C} := (C_i)_{i \in I}$.

We denote by $A^* := A \setminus \{\vec{C}\}$ the set of all action profiles in which at least one player quits, by $A_1^* := \{(Q_i, C_{-i}) : i \in N\}$ the set of all action profiles in which exactly one player quits, where $C_{-i} := (C_j)_{j \neq i}$, and by $A_{\geq 2}^* := A^* \setminus A_1^*$ the set of all action profiles in which at least two players quit. It is convenient to normalize the payoff function so that $r_i(Q_i, C_{-i}) = 0$ for every $i \in I$.

A *mixed action profile* is a vector $\xi = (\xi_i)_{i \in I} \in [0,1]^I$, with the interpretation that ξ_i is the probability with which player i quits. The probability of absorption under the mixed action profile ξ is $p(\xi) := 1 - \prod_{i \in I}(1 - \xi_i)$. Extend the absorbing payoff to mixed action profiles that are absorbing with positive probability: for every $\xi \in [0,1]^I$ such that $p(\xi) > 0$, define $r(\xi) := \frac{\sum_{a \in A^*} \xi(a) r(a)}{p(\xi)}$, where $\xi(a) := \left(\prod_{\{i :\, a_i = Q_i\}} \xi_i\right) \cdot \left(\prod_{\{i :\, a_i = C_i\}} (1 - \xi_i)\right)$, for every $a \in A$.

A (behavior) *strategy* of player i is a function $x_i = (x_i^n)_{n \in \mathbb{N}} : \mathbb{N} \to [0,1]$, with the interpretation that x_i^n is the probability that player i quits at stage n if the game did not terminate before that stage. A *strategy profile* is a vector $x = (x_i)_{i \in I}$ of strategies, one for each player.

Denote by $\theta := \inf\{n \in \mathbb{N} \colon a^n \in A^*\}$ the stage of termination; $\theta = \infty$ if all players continue throughout the game. Every strategy profile x induces a probability distribution $\mathbf{P}_x$ over the set of histories. Denote by $\mathbf{E}_x$ the corresponding expectation operator. A strategy profile x is *absorbing* if $\mathbf{P}_x(\theta < \infty) = 1$.

The *payoff* under strategy profile x is

$$\gamma(x) := \mathbf{E}_x \left[\mathbf{1}_{\{\theta < \infty\}} r(a^\theta) + \mathbf{1}_{\{\theta = \infty\}} r(\vec{C}) \right].$$

Let $\varepsilon \geq 0$. A strategy profile x^* is an *ε-equilibrium* if $\gamma_i(x^*) \geq \gamma_i(x_i, x^*_{-i}) - \varepsilon$ for every player $i \in I$ and every strategy x_i of player i. A strategy profile x^* is a *subgame-perfect ε-equilibrium* if for every $n \in \mathbb{N}$, the strategy profile $(x^{*,n}, x^{*,n+1}, \dots)$ is an ε-equilibrium. When $r_i(\vec{C}) < r_i(Q_i, C_{-i}) = 0$ for some $i \in I$, any subgame-perfect ε-equilibrium is *absorbing*, provided that ε is small enough. A payoff vector $w \in \mathbb{R}^I$ is a *subgame-perfect equilibrium payoff* if $w = \lim_{\varepsilon \to 0} \gamma(x^\varepsilon)$, where x^ε is a subgame-perfect ε-equilibrium for every $\varepsilon > 0$, and it is said to be *absorbing* if x^ε is an absorbing strategy profile for every $\varepsilon > 0$.

Three-player quitting games admit equilibrium payoffs (Solan, 1999).[3] It is not known whether the same applies to quitting games with at least four players. Sufficient conditions that guarantee the existence of a subgame-perfect equilibrium payoff have been provided by Solan and Vieille (2002), Simon (2007, 2012), Solan and Solan (2020), and AKRS. The latter defines a novel concept of *absorption paths*, which is a way to represent strategy profiles, and relates them to the set of limits of absorbing subgame-perfect ε-equilibrium strategy profiles.

2.2 APS Approach

The APS approach is a technique to characterize the set of subgame-perfect equilibrium payoffs in discounted games as the largest fixed point of the Shapley operator.

Specifically, suppose that players discount their payoffs by a discount rate $\beta \in (0,1)$. For every vector $v \in \mathbb{R}^I$, let $G_\beta(v)$ be the strategic-form game, where the set of players is I, the set of actions of each player $i \in I$ is A_i, the payoff if at least one player quits is given by r, and the payoff if all players continue is $\beta r(\vec{C}) + (1-\beta)v$. Let $\mathbf{Sh}_\beta(v)$ be the set of all equilibrium payoffs in $G_\beta(v)$.

For every bounded set $E \subset \mathbb{R}^I$, denote

$$\mathbf{Sh}_\beta(E) := \bigcup_{v \in E} \mathbf{Sh}_\beta(v) \subset \mathbb{R}^I.$$

It follows from Fink (1964) that for *every* bounded subinvariant set E of $\mathbf{Sh}_\beta$, i.e., $E \subseteq \mathbf{Sh}_\beta(E)$, any payoff in E can be attained in a β-discounted subgame-perfect equilibrium of the quitting game Γ. Moreover, the set of *all* such payoffs is precisely the largest bounded subinvariant set of $\mathbf{Sh}_\beta$.

The APS approach to finding the largest bounded subinvariant set of $\mathbf{Sh}_\beta$ is to repeatedly apply the Shapley operator to some bounded set that is known to contain all of the β-discounted subgame-perfect equilibrium payoffs, e.g., the convex hull of $(r(a))_{a \in A^*}$ and $\vec{0}$ will do. Successive applications of $\mathbf{Sh}_\beta$ generate a decreasing sequence of sets that is guaranteed to converge to the set of β-discounted subgame-perfect equilibrium payoffs.

When trying to adapt this approach to undiscounted games, one may be tempted to consider for every $v \in \mathbb{R}^I$ the strategic-form game $G_0(v)$ and the operator $\mathbf{Sh}_0$, which are defined as $G_\beta(v)$ and $\mathbf{Sh}_\beta$ for $\beta = 0$. It is true that the set of all subgame-perfect

[3] In fact, using arguments from Solan (1999) and Solan and Vohra (2001), it can be shown that three-player quitting games admit subgame-perfect equilibrium payoffs.

equilibrium payoffs is subinvariant with respect to $\mathbf{Sh}_0$, but it might be different from the largest bounded subinvariant set. As a result, the output of the APS approach may contain vectors that are not subgame-perfect equilibrium payoff of the quitting game Γ. To illustrate this point, consider the following example, due to Flesch, Thuijsman, and Vrieze (1997).

Example 2.2 (Flesch, Thuijsman, and Vrieze (1997)) *Consider the three-player quitting game where the payoff function r is given by the table in Figure 1.*

C^3

	C^2	Q^2
C^1	$-1,-1,-1$	$-1,0,2$
Q^1	$0,2,-1$	$0,-1,0$

Q^3

	C^2	Q^2
C^1	$2,-1,0$	$0,0,-1$
Q^1	$-1,0,0$	$-1,-1,-1$

Figure 1: The three-player game in Example 2.2.

We note that $r_i(\vec{C}) = -1 < 0 = r_i(Q_i, C_{-i})$ for all $i \in I$, and therefore every subgame-perfect ε-equilibrium must be absorbing, for all $\varepsilon < 1$.

Flesch, Thuijsman, and Vrieze (1997) proved that the set of subgame-perfect equilibrium payoffs in this example coincides with the boundary of the triangle whose extreme points are $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$. For example, one subgame-perfect equilibrium is as follows: Player 1 (resp. Player 2, Player 3) quits with probability $\frac{1}{2}$ in stages $t = 0 \mod 3$ (resp. $t = 1 \mod 3$, $t = 2 \mod 3$). In fact, all subgame-perfect equilibria have such a cyclic nature: the play is divided into countably many blocks, in each block, a single player quits with positive probability, the total probability that that player quits in the block is $\frac{1}{2}$ (except in the first block, where this probability is at most $\frac{1}{2}$), and if player i quits in some block, then player $(i+1) \mod 3$ quits in the following block.

To see that the APS approach does not apply to the undiscounted game, note that if v satisfies $v_i \geq 0$ for each $i \in I$, then $v \in \boldsymbol{Sh}_0(v)$. Thus, all vectors v in the convex hull of $(r(a))_{a \in A}$ that lie in the positive orthant satisfy $v \in \boldsymbol{Sh}_0(v)$. Yet, by Flesch, Thuijsman, and Vrieze (1997), they are not equilibrium payoffs.

The adaptation of the APS approach to undiscounted games is not straightforward, as it presents two hurdles.

First, while the APS approach characterizes discounted equilibria, in an undiscounted game, 0-equilibria need not exist, and the goal is to characterize the limit set of ε-equilibrium payoffs as ε goes to 0. To overcome this difficulty, we will use a new representation of strategy profiles that was recently developed in AKRS. In this representation, limits of ε-equilibria are 0-equilibria in a proper sense. This representation will be motivated in Section 2.3 and formally defined in Section 2.4.

The second hurdle is the one expressed by the example above that the largest invariant set of the Shapley operator may be strictly larger than the set of undiscounted equilibrium payoffs. To overcome this difficulty, we will restrict attention to a certain subset of the set of all undiscounted equilibrium payoffs, and we will use properties of the payoff function of the game that are summarized by a properly defined graph, see Section 4.1.

These two aspects are rectified by our approach, as will be exhibited by several examples in Section 4.4.

2.3 Motivating Example

Suppose that $(x^\varepsilon)_{\varepsilon>0}$ is a sequence of absorbing subgame-perfect ε-equilibria, and suppose w.l.o.g. that $w = \lim_{\varepsilon\to 0}\gamma(x^\varepsilon)$ exists. To study subgame-perfect equilibrium payoffs, one is tempted to study the limit strategy profile $(x^{0,n})_{n\in\mathbb{N}}$ that is defined by $x^{0,n} := \lim_{\varepsilon\to 0} x^{\varepsilon,n}$ for every $n \in \mathbb{N}$ (assuming this limit exists). If $(x^{0,n})_{n\in\mathbb{N}}$ is absorbing, then, since (x^ε) are subgame-perfect ε-equilibria for every $\varepsilon > 0$, it follows from the results in Vrieze and Thuijsman (1989) or Solan (1999) that it is a subgame-perfect 0-equilibrium. However, it might happen that while going to the limit as ε goes to 0, some probability of absorption is lost, and then the limit $(x^{0,n})_{n\in\mathbb{N}}$ is not necessarily a subgame-perfect 0-equilibrium (or even an ε-equilibrium, for $\varepsilon > 0$ sufficiently small). In fact, the following example shows that even the limit of subgame-perfect 0-equilibria is not necessarily an ε-equilibrium.

Example 2.2: continued.

Consider the three-player quitting game from Example 2.2.

Recall that $r_i(\vec{C}) = -1 < 0 = r_i(Q_i, C_{-i})$ *for all* $i \in I$*, and therefore every subgame-perfect* ε*-equilibrium must be absorbing, for all* $\varepsilon < 1$.

For each $k \geq 1$*, let* $\delta^k := 1 - (\frac{1}{2})^{1/k}$*, so that* $\prod_{n=1}^{k}(1-\delta^k) = \frac{1}{2}$*. Consider the strategy profile* x^k*, which repeats the following block of* $3k$ *action profiles:*

- *In stages* $0, 1, \ldots, k-1$ *the players play* $(\delta^k, 0, 0)$.
- *In stages* $k, k+2, \ldots, 2k-1$ *the players play* $(0, \delta^k, 0)$.
- *In stages* $2k, 2k+2, \ldots, 3k-1$ *the players play* $(0, 0, \delta^k)$.

By Flesch, Thuijsman, and Vrieze (1997), x^k *is a subgame-perfect* 0*-equilibrium for every* $k \geq 1$*. Since* $\delta^k \searrow 0$*, the limit of* x^k *as* k *goes to* ∞ *is the strategy profile in which all players always continue, which is not an* ε*-equilibrium for all* $\varepsilon \in [0,1)$*.*[4]

To overcome these weaknesses in the concept of equilibria, AKRS presented the concept of absorption paths, which does not allow the probability of absorption to be lost when taking the limit as ε goes to 0. More precisely, ARKS shows that by re-parameterizing time (originally *equal to* $\mathbb{N}$) by the nondecreasing probability of absorption, the set of all strategy profiles can be embedded in a sequentially compact set of continuous-time, A^*-valued paths, that contains also the limit case where players quit during some determined times with infinitesimal probabilities. AKRS prove that w is an absorbing subgame-perfect 0-equilibrium payoff if and only if there exists an absorption path satisfying a certain notion of subgame perfectness and w is the expected payoff under this absorption path.

[4] In a slight modification of Example 2.2, there are subgame-perfect ε-equilibria for all $\varepsilon > 0$ but no 0-equilibrium, see Solan (2001).

We refer to ARKS for an extensive treatment of this approach. Here, we focus on strategy profiles where at each time only one player quits with vanishing probability, like in Example 2.2. We shall therefore define a subclass of absorption paths, where only continuous-time aspects appear and where at every time instance, at most one player quits with a positive rate. The main result of the paper is an APS-like characterization of the set of absorbing subgame-perfect equilibrium payoffs that can be attained by such absorption paths.

In Example 2.2, for each $k \in \mathbb{N}$, the strategy profile x^k can be described as follows: for each $n \geq 1$,

- set $t^{k,n} := \mathbf{P}_{x^k}(\theta < n)$,
- set $\iota^k(t) := i$ whenever $t \in \left[t^{k,n}, t^{k,n+1}\right)$ and $x_i^{k,n}$, player i's probability of quitting under x^k at stage n, is positive.

The functions $(\iota^k)_{k\in\mathbb{N}}$, as well as their limit as k tends to $+\infty$, are all equal to the function ι that is piecewise constant and equal to $n+1 \mod (3)$ in the interval $[1-\frac{1}{2^n}, 1-\frac{1}{2^{n+1}})$, for every $n \in \mathbb{N}$.

More generally, in the following section, we introduce a set of right-continuous I-valued functions, which we call *Flesch absorption paths.* A Flesch absorption path is a special case of absorption paths where only continuous-time aspects appear, and where at every time instance, at most one player quits with a positive rate.

In this new setting, we define a natural notion of absorbing subgame-perfect equilibrium payoffs and show that they can be characterized by an APS-like approach. This characterization can also be used to compute them explicitly, and it induces a characterization of the corresponding set of subgame-perfect ε-equilibria in the standard setting (see Theorem 4.15 in AKRS).

In Section 5.2, we show how our approach can be used to go beyond the class of Flesch absorption paths and characterize the set of absorbing subgame-perfect equilibrium payoffs that can be attained by continuous absorption paths, i.e., those in which multiple players quit in continuous time throughout the play.

2.4 Flesch Absorption Paths

Definition 2.3 *A Flesch Absorption Path (FAP)*[5] *is a right-continuous map $\iota : [0,1) \to I$. We denote by $H(\iota)$ the set of its discontinuities, and remark that, because of the right continuity, $H(\iota)$ is well-ordered and finite or countable infinite.*

The interpretation of an FAP is as follows. Quitting occurs in continuous time, and at every $t \in [0,1)$, a single player $\iota(t)$ quits at rate one. The parameter t does not represent time, but rather the total probability of absorption. For each $t \in H(\iota)$, if $H(\iota) \cap (t,1) = \emptyset$, then player $\iota(t)$ is the last to quit; and, if $H(\iota) \cap (t,1) \neq \emptyset$, then there exists another player

[5]The reason for the appellation is that the equilibria in Example 2.2, which was presented and studied by Flesch, Thuijsman, and Vrieze (1997) and has greatly influenced the study of stochastic games since its publication, can be presented by this type of absorption paths and, in fact, motivated this concept.

who quits after player $\iota(t)$. In the latter case, we shall call the minimal element in $H(\iota)$ larger than t, which exists since $H(\iota)$ is well ordered, the *successor* of i.

Example 2.4 *The FAP ι defined by $\iota([0,1)) = \{1\}$ corresponds to the behavior that Player 1 is the only player who quits. The FAP that corresponds to a situation where players 1 and 2 alternately quit in continuous time, each with probability $\frac{1}{2}$, is given by*

$$\iota(t) = \begin{cases} 1, & t \in [0, \frac{1}{2}) \cup [\frac{1}{4}, \frac{1}{8}) \cup \cdots, \\ 2, & t \in [\frac{1}{2}, \frac{1}{4}) \cup [\frac{1}{8}, \frac{1}{16}) \cup \cdots. \end{cases}$$

More generally, suppose that $H(\iota) = (t^n)_{n \in \mathbb{N}}$, where $0 = t^0 < t^1 < \cdots$ and $\lim_{n \nearrow \infty} t^n = 1$. This FAP corresponds to a situation where the first player $\iota(t^0)$ quits with probability $\frac{t^1 - t^0}{1 - t^0} = t^1$, then; if the game hasn't terminated, player $\iota(t^1)$ quits with probability $\frac{t^2 - t^1}{1 - t^1}$, and so on.

Remarks 2.5 *Let ι be an FAP as defined above.*

1. *A point $t \in (0,1)$ (resp. $t \in [0,1)$) is a* left accumulation point *(resp.* right accumulation point*) of $H(\iota)$ if there is a sequence $(t^n)_{n \in \mathbb{N}}$ of points in $H(\iota)$ that strictly increases (resp. decreases) to t. Left accumulation points of $H(\iota)$ allow us to describe players' behavior where they quit with arbitrarily small probabilities. The assumption of right continuity excludes the existence of right accumulation points.*

2. *The assumption that ι is right-continuous guaranties that, for each connected component (t, t') of $[0,1) \setminus H(\iota)$, the value of ι at t is the same as on (t, t').*

3. *The set of FAPs is not compact in the weak topology of càdlàg paths. Indeed, one can devise a sequence of FAPs whose limit satisfies that two players quit simultaneously in continuous time. For example, for each $k = 2, 3, \ldots$, let ι^k be the FAP defined by $\iota^k(t) := tk + 1 \mod (2)$ so that $H(\iota^k) = \{0, \frac{1}{k}, \ldots, \frac{k-1}{k}\}$. The FAP ι^k corresponds to a situation where Player 1 quits with probability $\frac{1}{k}$, then Player 2 quits with probability $(\frac{1}{k})/(\frac{k-1}{k}) = \frac{1}{k-1}$, then Player 1 quits with probability $\frac{1}{k-2}$, and so on.*

   The limit of the sequence of these FAPs is the strategy profile in continuous time in which both players quit simultaneously throughout the game at the same rate. Yet, this behavior cannot be described by an FAP.

4. *For each $i \in I$, the total probability that player i quits in the interval $[a,b) \subset [0,1)$ is $\mathrm{Leb}(\{s \in [a,b) : \iota(s) = i\})$, where Leb is the Lebesgue measure. In particular, $\mathrm{Leb}(\{s \in [t,1) : \iota(s) = i\})$ for $t \in [0,1)$ is the total probability that player i quits after absorption probability t.*

We now define the expected payoff under an FAP.

Definition 2.6 *For every $t \in [0,1)$, the* expected payoff *after absorption probability t is:*

$$\gamma^t(\iota) := \sum_{i \in I} \frac{\mathrm{Leb}(\{s \in [t,1) : \iota(s) = i\})}{1-t} \cdot R_i, \tag{1}$$

where $R_i := r(Q_i, C_{-i})$ is the payoff when player i quits alone.

Since FAPs model behavior in continuous time, in which players control the rate at which they quit, players cannot quit simultaneously, and hence the payoff vector $\gamma_t(\iota)$ depends only on the payoffs when players quit alone. We let R be the $(|I| \times |I|)$-*payoff-matrix of single quittings* whose i'th row is R_i. As we normalized payoffs so that $R_{ii} = 0$ for each $i \in I$, the diagonal of R is $\vec{0} = (0, 0, \ldots, 0)$. Furthermore, we assume throughout the paper that the matrix R is *generic* in the following sense.

Assumption 2.7 (Genericity of payoffs) *The quitting game $\Gamma = (I, r)$ satisfies the following genericity assumption:*

$$R_{i,j} = 0 \iff i = j.$$

Recall that a strategy profile is a subgame-perfect ε-equilibrium if in any subgame no player can profit more than ε by deviating. When time is continuous and players control the rate at which they quit, players cannot quit simultaneously, hence this requirement translates into two conditions: *i*) a player who quits with a positive rate is indifferent between quitting and continuing, and *ii*) a player who quits with a rate of 0 cannot profit by quitting. This leads to the following definition of sequential perfectness for FAPs, which is adapted from AKRS.

Definition 2.8 (AKRS) *An FAP ι is* sequentially perfect *if for every $t \in [0, 1)$, $\gamma^t(\iota) \geq \vec{0}$ and $\gamma^t_i(\iota) = 0$ whenever $\iota(t) = i$.*

Remarks 2.9 *Let ι be an FAP.*

1. *Set $i = \iota(0)$ and suppose that $t = \inf\{s \geq 0 : \iota(s) \neq i\} < 1$. Then,*

$$\gamma^0(\iota) = tR_i + (1 - t)\gamma^t(\iota).$$

   In words, this equation says that the payoff (from 0 and on) is equal to the total probability that player $\iota(0)$ quits until time t, when another player gets to quit, times the payoff if player $\iota(0)$ quits, plus the probability that player $\iota(0)$ does not quit until time t, times the continuation payoff. A similar remark applies to any discontinuity point of ι, and not only to 0.

2. *Given $t \in [0, 1)$, the FAP induced by ι in the subgame starting at t is the FAP $\iota^t : [0, 1) \to I$ defined by $\iota^t(s) := \iota(t + (1 - t)s)$.*
   - *It holds that $\gamma^t(\iota) = \gamma^0(\iota^t)$.*
   - *If ι is sequentially perfect, then so is ι^t.*

3. *If ι is sequentially perfect, then $\gamma^0_{\iota(t)}(\iota^t) = 0$ for every $t \in [0, 1)$.*

Remark 2.10 *A sequentially perfect FAP does not necessarily correspond to an ε-equilibrium. Indeed, consider the quitting game with two players, where the payoff function is given by*

$$r(C_1, C_2) = (1, 1), \quad r(Q_1, C_2) = (0, 1), \quad r(C_1, Q_2) = (1, 0), \quad r(Q_1, Q_2) = (1, 1).$$

The FAP $\iota \equiv 1$ is sequentially perfect, but it is not an ε-equilibrium for $\varepsilon \in [0, 1)$, because by never quitting player 1 guarantees a payoff of 1.

Denote the *set of sequentially perfect FAPs* by Υ, and let $\mathcal{E}$ be the set of payoffs that can be attained by them, that is

$$\mathcal{E} := \left\{ w \in \mathbb{R}^I \ : \ \exists\, \iota \in \Upsilon \text{ s.t. } w = \gamma^0(\iota) \right\}.$$

We seek to characterize $\mathcal{E}$.[6]

3 APS Approach for FAPs

In this section, we present an analog of the APS approach to FAPs. Although this naive approach fails to characterize the set of sequentially perfect equilibrium payoffs, it will serve as a basis for the more sophisticated approach we will present in Section 4.

For every nonempty subset of players $N \subseteq I$, denote by $\mathcal{R}_N$ the set of nonnegative payoffs that can be generated by FAPs, not necessarily sequentially perfect, in which only players in N can ever quit, that is,

$$\mathcal{R}_N := \left\{ w \in \mathbb{R}^I_+ : \exists \text{ FAP } \iota \text{ s.t. } w = \gamma^0(\iota),\ \iota([0,1)) \subseteq N \right\} = co(\{R_i : i \in N\}) \cap \mathbb{R}^I_+.$$

It is convinient to define a subspace $\mathcal{H}_N$ of $\mathbb{R}^I$ in which every player in N receives exactly 0, i.e., $\mathcal{H}_N := \{w \in \mathbb{R}^I \colon w_i = 0\ \forall i \in N\}$.

Remark 3.1 *It is evident that $\mathcal{E} \subseteq \mathcal{R}_I$. On the other hand, for any $i \in I$, we have $\mathcal{R}_{\{i\}} \subseteq \mathcal{E}$. Indeed, if $R_i \in \mathbb{R}^I_+$, then the FAP ι with $\iota(t) = i$ for all $t \in [0,1)$ belongs to Υ; and, if $R_i \notin \mathbb{R}^I_+$, then $\mathcal{R}_{\{i\}} = \emptyset$ and the assertion trivially holds. In general, it is not true that $\mathcal{E} = \mathcal{R}_I$. For instance, consider the three-player game in Example 2.2. In this game, $\mathcal{R}_I$ is the triangle whose extreme points are $(0,0,1)$, $(0,1,0)$, and $(1,0,0)$, while $\mathcal{E}$ is the boundary of this triangle as shown in Flesch, Thuijsman, and Vrieze (1997).*

The following lemma recursively unpacks the set $\mathcal{E}$, stating that if $w \in \mathcal{E}$, then (i) $w_i = 0$ for some player i, and (ii) w is a convex combination of the payoff if player i quits alone and the payoff induced by some sequentially perfect FAP. This recursive representation will serve as a basis for our first approach.

Lemma 3.2

$$\mathcal{E} = \left\{ w \in \mathbb{R}^I_+ : \exists\, (\lambda, i, \iota) \in [0,1] \times I \times \Upsilon \ s.t.\ w = \lambda R_i + (1-\lambda)\gamma^0(\iota),\ w_i = 0 \right\}. \qquad (2)$$

Proof. Let $w \in \mathcal{E}$, i.e., $w = \gamma^0(\iota)$ for some sequentially perfect FAP ι. Taking $\lambda = 0$ and $i = \iota(0)$ in the right side of relation (2) makes it clear that it contains w.

Conversely, let $\lambda \in [0,1]$, $i \in I$, and $\iota \in \Upsilon$ such that $w = \lambda R_i + (1-\lambda)\gamma^0(\iota)$ and $w_i = 0$. If $\lambda = 0$, then $w = \gamma^0(\iota)$ for the sequentially perfect FAP ι, thus $w \in \mathcal{E}$. If $\lambda = 1$, then

[6] While the set $\mathcal{E}$ is generally a superset of the set of subgame-perfect equilibrium payoffs (see the example in Remark 2.10), the two sets coincide as soon as, e.g., $r(\vec{C})$ lies in the nonpositive orthant.

$w = R_i \geq \vec{0}$. It follows that w is attained by the FAP $\iota' \in \Upsilon$ with $\iota'(t) = i$ for all $t \in [0, 1)$. If $\lambda \in (0, 1)$, then define an FAP ι' by setting

$$\iota'(t) := \begin{cases} i, & \text{if } t \in [0, \lambda), \\ \iota\left(\frac{t-\lambda}{1-\lambda}\right), & \text{if } t \in [\lambda, 1). \end{cases}$$

Clearly, $\iota' \in \Upsilon$ and $w = \gamma^0(\iota')$, thus it follows that $w \in \mathcal{E}$. ■

In what follows, we shall construct the *APS operator* and show that the set $\mathcal{E}$ is invariant with respect to it. For each player $i \in I$ and every subset $E \subseteq \mathcal{R}_I$, define the set of payoffs $\mathbf{T}_i(E)$ that can be attained with continuation payoffs in E when player i is the first quitter, that is,

$$\begin{aligned} \mathbf{T}_i(E) :=& \ \left\{w \in \mathbb{R}^I_+ : \exists\, (\lambda, v) \in [0, 1] \times E \text{ s.t. } w = \lambda R_i + (1 - \lambda)v,\ w_i = 0\right\} \\ =& \ co\left(\{R_i\} \cup E\right) \cap \mathcal{H}_{\{i\}} \cap \mathbb{R}^I_+. \end{aligned} \tag{3}$$

Define also

$$\mathbf{T}(E) := \bigcup_{i \in I} \mathbf{T}_i(E).$$

We list below several useful properties of the operator $\mathbf{T}$.

Remarks 3.3 *For all $E \subseteq \mathcal{R}_I$, it holds that*

1. *$T(E) \subseteq \mathbb{R}^I_+$,*
2. *If E is closed (resp. compact), then $\mathbf{T}(E)$ is closed (resp. compact) as well.*
3. *$\mathbf{T}(E) \subseteq \mathbf{T}(E')$ for every E and E' such that $E \subseteq E' \subseteq \mathcal{R}_I$, i.e., $\mathbf{T}$ is monotone in the set-inclusion order.*
4. *Lemma 3.2 implies that the set $\mathcal{E}$ is invariant for the operator $\mathbf{T}$, i.e., $\mathcal{E} = \mathbf{T}(\mathcal{E})$.*

Since $\mathbf{T}$ is monotone and $\mathcal{E} \subseteq \mathcal{R}_I$ is invariant for $\mathbf{T}$, we can follow Abreu, Pearce, and Stacchetti (1986) and bound $\mathcal{E}$ from above by the *largest invariant set* $\overline{\mathcal{E}}$ with respect to $\mathbf{T}$ within $\mathcal{R}_I$. The existence of this largest invariant set is guaranteed by Knaster-Tarski's Theorem (Knaster (1928), Tarski (1955)), which also suggests an iterative procedure for computing it: $\overline{\mathcal{E}}$ can be obtained by repeatedly applying the operator $\mathbf{T}$ to $\mathcal{R}_I$, that is

$$\overline{\mathcal{E}} = \bigcap_{n=0}^{\infty} \mathbf{T}^{(n)}(\mathcal{R}_I), \tag{4}$$

where $\mathbf{T}^{(n)}$ is the n-th application[7] of the operator $\mathbf{T}$.

Unfortunately, as exhibited by Example 2.2, iterations in Eq. (4) always terminate in a single step with $\overline{\mathcal{E}} = \mathcal{R}_I$, because $\mathcal{R}_I$ is itself invariant for $\mathbf{T}$. As already illustrated by the aforementioned example, there is no reason to expect that all the payoffs in $\mathcal{R}_I$ are sequentially perfect equilibrium payoffs. In fact, it might happen that there is no sequentially perfect FAP, thus $\mathcal{E} = \emptyset$, but $\mathcal{R}_I \neq \emptyset$ as happens in the example below.

[7]Due to monotonicity of $\mathbf{T}$, repeated applications of this operator to $\mathcal{R}_I$ generate a nested nonincreasing sequence of sets, which yields an outer approximation to $\overline{\mathcal{E}}$, hence $\mathcal{E}$.

Example 3.4 *Consider the quitting game with four players and the following payoff matrix of single quittings, which is a variant of the game studied in Solan and Vieille (2002):*

$$R = \begin{pmatrix} 0 & 4 & -1 & -1 \\ -1 & 0 & 4 & 1 \\ -1 & -1 & 0 & 4 \\ 4 & 1 & -1 & 0 \end{pmatrix}.$$

The set $\mathcal{R}_I$ is the convex hull of $(0, \frac{5}{2}, 0, 0)$, $(0, 0, 0, \frac{5}{2})$, $(\frac{11}{9}, 0, 0, \frac{17}{9})$, $(0, 0, \frac{11}{5}, \frac{7}{5})$, $(0, \frac{17}{9}, \frac{11}{9}, 0)$, $(\frac{11}{5}, \frac{7}{5}, 0, 0)$, $(3, \frac{4}{5}, 0, \frac{1}{5})$ and $(0, \frac{1}{5}, 3, \frac{4}{5})$. However, the set $\mathcal{E}$ is empty. To see why this is true, we first argue that if there is a sequentially perfect FAP ι, then it cannot be that only one player quits along ι. Indeed, for every i, the vector R_i contains a negative entry, and the payoff along a sequentially perfect FAP must be in the nonnegative orthant.

We also claim that along such a sequentially perfect FAP ι, the players necessarily quit in a cyclic order. That is, for every $t \in [0,1)$, the successor of player $\iota(t)$ is player $(\iota(t)+1) \mod 4$. Indeed, if $\iota(t) = i$ and $\iota(t') = j$, then $\gamma_i^{t'}(\iota) = \gamma_j^{t'}(\iota) = 0$. Since $\gamma(\iota)$ lies in the nonnegative orthant, this implies that $R_{i,j} > 0$ and $R_{j,i} < 0$. However, the only pairs (i,j) that satisfy these inequalities are those that satisfy $j = (i+1) \mod 4$.

The exact numbers in R were selected in such a way that the negative payoffs are sufficiently high, so that there cannot be a sequentially perfect FAP in the nonnegative orthant. Below (see page 24) we formally show that $\mathcal{E}$ is indeed empty.

In summary, the naive APS approach, which is inspired by the classical recursive algorithm of Abreu, Pearce, and Stacchetti (1986), is not applicable in the undiscounted setting. Even though the set $\mathcal{E}$ is invariant for $\mathbf{T}$, it might differ from the largest invariant sets $\overline{\mathcal{E}}$. To the best of our knowledge, there is no method that can be used to compute some invariant sets of $\mathbf{T}$, except the largest (and smallest) ones. The observation made in Example 3.4, namely that for two consecutive quitters i, j in every sequentially perfect FAP, necessarily $R_{i,j} > 0$ and $R_{j,i} < 0$ (when payoffs are generic), will be developed in the following section to adapt the APS approach to our setup.

4 Essential APS Approach

In this section, we adapt the APS operator to undiscounted quitting games. We start in Section 4.1 by studying the set of pairs of players (i,j) such that j can be a successor of i in a sequentially perfect FAP. In Section 4.2, we construct an operator, which we term the *essential APS* operator, and show that the set of sequentially perfect FAP payoffs is invariant under this operator. In particular, the largest invariant set of this operator contains the set of sequentially perfect FAP payoffs. In Section 4.3, we study some properties of the essential APS operator and provide a sufficient condition that ensures that the largest invariant set of this operator coincides with the set of sequentially perfect FAP payoffs. Section 4.4 provides several numerical examples that illustrate the essential APS operator and shows that this operator may be useful in approximating the set of sequentially perfect FAP payoffs.

4.1 Graph of Play

Recall that every FAP ι indicates for each time instance a player who quits with a positive rate. The following lemma shows that not all quitting orders are compatible with sequential perfectness. Specifically, it establishes that the *set of player i's potential successors* S_i is given by

$$S_i := \{j \in I : R_{j,i} < 0 < R_{i,j}\}. \tag{5}$$

Lemma 4.1 *Let ι be a sequentially perfect FAP with $\iota(0) = i$. The following three claims hold:*

(C1) There exists $j \neq i$ such that $0 < R_{i,j}$.

(C2) Let $t \in [0,1)$. If $t' = \inf\{s \geq t : \iota(s) \neq \iota(t)\} < 1$, then we have

- *(a) $t'' = \inf\{s > t' : \iota(s) \neq \iota(t')\} < 1$,*
- *(b) $R_{\iota(t'),\iota(t)} < 0 < R_{\iota(t),\iota(t')}$, i.e., $\iota(t') \in S_{\iota(t)}$.*

(C3) For each left accumulation point $t \in (0,1)$, there exists $j \in I$ such that $\iota(t) \in S_j$.

Proof. Let us prove (C1). There are two cases to consider, namely $\iota([0,1)) = \{i\}$ and $\iota([0,1)) \neq \{i\}$. Suppose first that $\iota([0,1)) = \{i\}$. Then, $\gamma^0(\iota) = R_i$. Since ι is sequentially perfect, we must have $R_i \geq \vec{0}$. Claim (C1) follows from Assumption 2.7 on the payoff matrix of single quittings.

Suppose next that $\iota([0,1)) \neq \{i\}$. Since $H(\iota)$ is well-ordered, a different player gets to quit at $t = \inf\{s \geq 0 : \iota(s) \neq i\} > 0$. By Remark 2.9.1,

$$\gamma^0(\iota) = tR_i + (1-t)\gamma^t(\iota).$$

Let $j = \iota(t)$. Sequential perfectness asks for $\gamma_j^0(\iota) \geq 0$ and $\gamma_j^t(\iota) = 0$. By Assumption 2.7, we necessarily have $R_{i,j} > 0$, which proves (C1).

We now show (C2) for $t = 0$ and $t' < 1$. Then (a) is proven as soon as we show that $\iota([t',1)) \neq \{j\}$. Indeed, if $\iota([t',1)) = \{j\}$, then $\gamma^{t'}(\iota) = R_j$. In particular, we have $R_{j,i} = \gamma_i^{t'}(\iota) = 0$, which contradicts Assumption 2.7. It follows from the same argument as above that another player gets to quit at $t'' < 1$, and

$$(1-t')\gamma^{t'}(\iota) = (t''-t')R_j + (1-t'')\gamma^{t''}(\iota).$$

Since ι is sequentially perfect, we have $\gamma_i^{t'}(\iota) = 0$ and $\gamma_i^{t''}(\iota) \geq 0$. It follows that $R_{j,i} < 0$, and therefore $j \in S_i$.

Let now t be an arbitrary element of $[0,1)$ with $t' \neq 1$. Since the FAP ι^t, which is defined in Remark 2.9.2, is sequentially perfect, we must have again that $t'' < 1$ and $\iota(t') \in S_{\iota(t)}$.

Finally let us show that (C3) holds. By Definition 2.6 and Remarks 2.9, for every $t' \in [0,t)$, the players' payoffs under ι can be written as follows:

$$(1-t')\gamma^{t'}(\iota) = \sum_{j \in I} Leb(\{s \in [t',t) : \iota(s) = j\}) \cdot R_j + (1-t)\gamma^t(\iota).$$

It follows that $\gamma^{t'}(\iota)$ converges to $\gamma^t(\iota)$ as t' goes to t. Since, by Definition 2.8, $\gamma^{t'}_{\iota(t')}(\iota) = 0$ for all $t' \in [0, t)$, we necessarily have that $\gamma^t_j(\iota) = 0$ for all $j \in \bigcap_{t' \in [0,t)} \iota([t', t)) \setminus \{\iota(t)\}$. We claim that $\iota(t)$ is a successor of at least one such player j.

Indeed, by the same argument as above, $R_{\iota(t),j} < 0$ for each such player $j \in \bigcap_{t' \in [0,t)} \iota([t', t)) \setminus \{\iota(t)\}$. On the other hand, if $R_{j,\iota(t)} < 0$ for all $j \in \bigcap_{t' \in [0,t)} \iota([t', t)) \setminus \{\iota(t)\}$, then for every t' sufficiently close to t,

$$\sum_{j \in I} Leb(\{s \in [t', t) : \iota(s) = j\}) \cdot R_{j,\iota(t)} + (1 - t)\gamma^t_{\iota(t)}(\iota) < 0,$$

which contradicts sequential perfectness of ι. ■

The following corollary follows directly from Lemma 4.1(C2)(a) applied to $t = 0$.

Corollary 4.2 *For any sequentially perfect FAP ι, the set of discontinuities $H(\iota)$ is empty or infinite.*

It is convenient to visualize the order of players' quittings by a *directed graph* (I, L), where I is the set of vertices (players) and $L := \{(i, j) \in I^2 : j \in S_i\}$ is the set of directed edges. To make further progress, we explore the topology of (I, L).

Let us introduce some auxiliary definitions. For each nonempty subset $N \subseteq I$, the *subgraph* (N, L_N) is the directed graph with set of vertices N and with set of directed edges $L_N := \{(i, j) \in N^2 : j \in S_i\}$. A *directed path in* (N, L_N) *from* $i \in N$ *to* $j \in N$ is a vector of distinct vertices $(i^1, ..., i^m) \in N^m$ such that $i^1 = i$, $i^m = j$, and $i^n \in S_{i^{n-1}}$ for $n = 2, ..., m$. We call the subgraph (N, L_N) a *simple circuit* if $|N| \geq 2$ and for all distinct $i, j \in N$ there exists a unique directed path in (N, L_N) from i to j.

A subgraph (N, L_N) is called a *strongly connected component* if N is a maximal set of vertices, such that in (N, L_N) there is a directed path between each distinct pair of these vertices. To simplify notations we identify a subgraph with its vertices, i.e., we write N instead of (N, L_N). Let $\mathbb{I}$ be the *set of strongly connected components of* (I, L). For each $N \in \mathbb{I}$, we denote by $\widehat{N} \subseteq I$ the set of all vertices not in N that are reachable from N:

$$\widehat{N} := \{j \in I \setminus N : \exists \text{ directed path in } I \text{ from some } i \in N \text{ to } j\}.$$

Remarks 4.3

1. *The set $\mathbb{I}$ is a partition of I, thus $\widehat{N}$ is the union of all strongly connected components disjoint of N that are reachable from N.*

2. *Each $N \in \mathbb{I}$ cannot consist of exactly two elements because for every distinct $i, j \in I$ we cannot have simultaneously $j \in S_i$ and $i \in S_j$, see Eq. (5).*

3. *The set of strongly connected components of (I, L) forms an acyclic graph. This is the graph whose vertices are the strongly connected components of (I, L), and there is a directed edge from the component N to the component N' if and only if $N' \subseteq \widehat{N}$.*

4. *Every player $i \in I$ with $R_i \leq \vec{0}$ forms a singleton strongly connected component; moreover, $\widehat{\{i\}} = \emptyset$.*

5. *Every player $i \in I$ with $R_i \geq \vec{0}$ forms a singleton strongly connected component; moreover, there is no $N \in \mathbb{I}$ such that $i \in \widehat{N}$.*

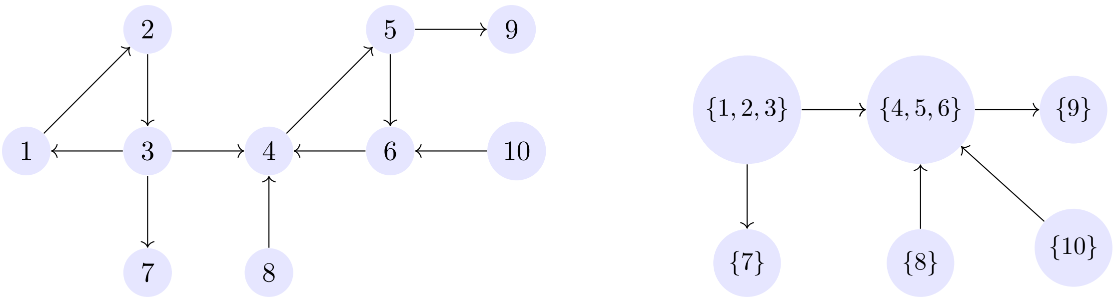
6. *Since the number of strongly connected components is finite, and they form an acyclic graph, there always exists at least one strongly connected component $N \in \mathbb{I}$ with $\widehat{N} = \emptyset$.*

7. *For every $N \in \mathbb{I}$ and every $i \in N$, we have $S_i \subseteq N \cup \widehat{N}$.*

Example 4.4 *Consider the following payoff matrix of single quittings:*

$$R = \begin{pmatrix} 0 & + & - & \times & \times & \times & \times & \times & - & + \\ - & 0 & + & \times & \times & \times & \times & \times & - & + \\ + & - & 0 & + & \times & \times & + & \times & - & + \\ \times & \times & - & 0 & + & - & \times & - & - & + \\ \times & \times & \times & - & 0 & + & \times & \times & + & + \\ \times & \times & \times & + & - & 0 & \times & \times & - & - \\ \times & \times & - & \times & \times & \times & 0 & \times & - & + \\ \times & \times & \times & + & \times & \times & \times & 0 & - & + \\ - & - & - & - & - & - & - & - & 0 & + \\ + & + & + & + & + & + & + & + & + & 0 \end{pmatrix},$$

where "+" ("-") stands for a positive (negative) entry, "$\times$" at positions (i, j) and (j, i) means that neither $i \in S_j$ nor $j \in S_i$, i.e., $R_{i,j} R_{j,i} > 0$. Figure 1 plots the directed graph (I, L). There are six strongly connected components, i.e., $\mathbb{I}$ consists of $\{1, 2, 3\}$, $\{4, 5, 6\}$,



(a) The graph over I.

(b) The graph over $\mathbb{I}$.

Figure 1: Graph of play in Example 4.4.

$\{7\}$, $\{8\}$, $\{9\}$, $\{10\}$. *As revealed by Figure 1,* $\widehat{\{7\}} = \widehat{\{9\}} = \emptyset$, $\widehat{\{4, 5, 6\}} = \{9\}$, $\widehat{\{8\}} = \widehat{\{10\}} = \{4, 5, 6, 9\}$, *and* $\widehat{\{1, 2, 3\}} = \{4, 5, 6, 7, 9\}$.

4.2 Constructing the Essential APS Operator

In this section, we build on the classical APS operator to construct a tighter bound for the set $\mathcal{E}$ of payoffs attainable by sequentially perfect FAPs. The gist of our construction is to inductively find the largest invariant sets in each strongly connected component of the graph (I, L) following their order prescribed by reachability. For this purpose, we shall use Lemma 4.5 to decompose $\mathcal{E}$. Recall that each set of payoffs $\mathcal{R}_{\{i\}} := \{R_i\} \cap \mathbb{R}^I_+$ corresponds to the payoff attained by the constant FAP $\iota \equiv i$, when it is sequentially perfect, and is otherwise empty.

Lemma 4.5 *It holds that* $\mathcal{E} = \bigcup_{i \in I} \mathcal{E}_i$*, where, for all* $i \in I$*,*

$$\mathcal{E}_i := \mathcal{R}_{\{i\}} \cup \Big\{ w \in \mathbb{R}^I_+ : \exists (\lambda, \iota) \in [0,1] \times \Upsilon \text{ s.t. } w = \lambda R_i + (1-\lambda)\gamma^0(\iota),\ w_i = 0,\ \iota(0) \in S_i \Big\}.$$

The set $\mathcal{E}_i$ in Lemma 4.5 contains all payoffs that can be attained by sequentially perfect FAPs in which player i receives 0, quits at the outset with some probability, and any player who quits next (if any) must be reachable from i in the graph (I, L).

Proof. Remark 3.1 and Lemma 3.2 imply that $\mathcal{E}_i \subseteq \mathcal{E}$ for all $i \in I$.

Conversely, for $w \in \mathcal{E}$, let $\iota \in \Upsilon$ be such that $w = \gamma^0(\iota)$, and $i = \iota(0)$. By Remark 4.3.7, $w_i = 0$. If $\iota([0,1)) = \{i\}$, then $w = R_i \in \mathcal{R}_{\{i\}} \subseteq \mathcal{E}$. If $\iota([0,1)) \neq \{i\}$, then, since ι is right-continuous, by Lemma 4.1, the value of $t = \inf\{s \geq 0 : \iota(s) \neq i\} \in (0,1)$ is positive and $\iota(t) \in S_i$. The assertion of the claim then follows from Remark 2.9.2: $\gamma^0(\iota) = tR_i + (1-t)\gamma^0(\iota^t)$ with $\iota^t \in \Upsilon$. ■

Recall that the naive APS operator $\mathbf{T}$ maps subsets of $\mathcal{R}_I$ to subsets of $\mathcal{R}_I$, and is defined so that the set $\mathcal{E}$ is invariant under it. In contrast, the *Essential APS* operator will take as input a collection of subsets of $\mathcal{R}_I$, outputs a collection of subsets of $\mathcal{R}_I$, and will be defined in such a way that $(\mathcal{E}_i)_{i \in I}$ is invariant under this operator. This operator will be used separately on each strongly connected component of (I, L), and it will be defined recursively along the graph of strongly connected components.

Let us fix some $N \in \mathbb{I}$ and $i \in N$. As mentioned in Remark 4.3.7, $S_i \subseteq N \cup \widehat{N}$. Given an arbitrary collection of sets indexed by elements of $\widehat{N}$, say $(E_j)_{j \in \widehat{N}} \subseteq (\mathcal{R}_I)^{\widehat{N}}$, the following operator is well-defined: for all collections $(E_j)_{j \in N} \subseteq (\mathcal{R}_I)^N$, set

$$\mathbf{F}_{N,i}\left((E_j)_{j \in N} | (E_j)_{j \in \widehat{N}}\right) := \mathcal{R}_{\{i\}} \cup \mathbf{T}_i\left(\bigcup_{j \in S_i} E_j\right) \subseteq \mathbb{R}^I_+. \tag{6}$$

The set $\mathbf{F}_{N,i}\left((E_j)_{j \in N} | (E_j)_{j \in \widehat{N}}\right)$ contains all payoffs that can be attained at time 0 when player i quits with some nonnegative probability, selects a player j in S_i and a continuation payoff in E_j that yields 0 to player i. To end up again with a collection, it is convenient to stack together $\mathbf{F}_{N,i}$ as

$$\mathbf{F}_N\left((E_i)_{i \in N} | (E_i)_{i \in \widehat{N}}\right) := \left(\mathbf{F}_{N,i}\left((E_j)_{j \in N} | (E_j)_{j \in \widehat{N}}\right)\right)_{i \in N} \subseteq \mathbb{R}^{I \times N}_+.$$

We next mention a few useful properties of $\mathbf{F}_N$.

Remarks 4.6 *Fix* $N \in \mathbb{I}$ *and* $(E_i)_{i \in N \cup \widehat{N}} \subseteq (\mathcal{R}_I)^{N \cup \widehat{N}}$.

1. *If the sets* $(E_i)_{i\in N\cup\widehat{N}}$ *are closed (resp. compact), then* $\mathbf{F}_N\left((E_i)_{i\in N}|(E_i)_{i\in\widehat{N}}\right)$ *is a collection of closed (resp. compact) sets.*

2. *For each* $i\in N$*, we have* $\mathbf{F}_{N,i}\left((E_j)_{j\in N}|(E_j)_{j\in\widehat{N}}\right)\subseteq\mathbf{F}_{N,i}\left((E'_j)_{j\in N}|(E'_j)_{j\in\widehat{N}}\right)$ *for every* $(E_j)_{j\in N\cup\widehat{N}}$ *and* $(E'_j)_{j\in N\cup\widehat{N}}$ *such that* $E_j\subseteq E'_j\subseteq\mathcal{R}_I$ *for all* $j\in N\cup\widehat{N}$*.*

3. *Lemmas 4.1 and 4.5 imply that the collection* $(\mathcal{E}_i)_{i\in N}$ *is a fixed point of the operator* $\mathbf{F}_N\left(\cdot|(\mathcal{E}_i)_{i\in\widehat{N}}\right)$*, that is*

$$(\mathcal{E}_i)_{i\in N}=\mathbf{F}_N\left((\mathcal{E}_i)_{i\in N}|(\mathcal{E}_i)_{i\in\widehat{N}}\right). \tag{7}$$

4.3 Characterization Based on the Essential APS Operator

By Remark 4.3.7, after a transition out of a strongly connected component N, the play will never return to it, hence we can construct the largest invariant sets $(\overline{\mathcal{E}_i})_{i\in I}$ of the Essential APS operator by induction, starting with all sets $N\in\mathbb{I}$ such that $\widehat{N}=\emptyset$, followed by the components $N\in\mathbb{I}$, such that $\widehat{N}\subseteq\cup_{N':\widehat{N'}=\emptyset}N'$, and so on. These largest invariant sets will be upper bounds of the sets $(\mathcal{E}_i)_{i\in I}$.

• Consider first $N\in\mathbb{I}$ such that $\widehat{N}=\emptyset$. Set $\mathcal{F}_N:=\mathcal{R}_N$, and define $(\overline{\mathcal{E}_i})_{i\in N}$ as follows:

$$(\overline{\mathcal{E}}_i)_{i\in N}:=\bigcap_{n=0}^{\infty}\mathbf{F}_N^{(n)}\left((\mathcal{F}_N)^N\mid\emptyset\right). \tag{8}$$

• Consider next $N\in\mathbb{I}$ such that $\widehat{N}\neq\emptyset$, and suppose that the sets $(\overline{\mathcal{E}}_i)_{i\in\widehat{N}}$ are already known. Let $\mathcal{F}_N$ be given by

$$\mathcal{F}_N:=co\left(\bigcup_{i\in N}\left(\left(\{R_i\}\cup\bigcup_{j\in S_i\cap\widehat{N}}\overline{\mathcal{E}}_j\right)\right)\cap\mathcal{H}_{\{i\}}\right)\cap\mathbb{R}^I_+. \tag{9}$$

The set $\mathcal{F}_N$ is the convex hull of the set of payoffs that can be attained by having some player $i\in N$ quit with a non-negative probability, and having the continuation payoff be taken from the upper bound $\overline{\mathcal{E}}_j$ of some successor j of i, provided i's continuation payoff is 0. We can see recursively that, for all $N\in\mathbb{I}$, $\mathcal{F}_N$ is a compact subset of $\mathcal{R}_I$. Then again, we can define the sets $(\overline{\mathcal{E}}_i)_{i\in N}$ by

$$(\overline{\mathcal{E}}_i)_{i\in N}:=\bigcap_{n=0}^{\infty}\mathbf{F}_N^{(n)}\left((\mathcal{F}_N)^N\mid(\overline{\mathcal{E}}_i)_{i\in\widehat{N}}\right). \tag{10}$$

The following lemma unpacks the sets $(\overline{\mathcal{E}}_i)_{i\in I}$ as defined in (8) and (13), and gives a first link between $(\overline{\mathcal{E}}_i)_{i\in I}$ and $\mathcal{E}$.

Lemma 4.7 *Let $N \in \mathbb{I}$. For each $i \in N$ and every $w \in \overline{\mathcal{E}}_i$, there exists a sequence $(w^n, i^n, \lambda^n)_{n\in\mathbb{N}} \subset (\mathbb{R}^I \times (N \cup \widehat{N}) \times [0,1])^{\mathbb{N}}$ with $w^0 = w$ and $i^0 = i$, such that for each $n \in \mathbb{N}$,*

$$\begin{cases} w^n \in \overline{\mathcal{E}}_{i^n}, \\ w^n = \lambda^n R_{i^n} + (1-\lambda^n) w^{n+1}, \\ i^{n+1} \in S_{i^n} \ \ \textit{whenever} \ \ \lambda^n \neq 1. \end{cases} \tag{11}$$

Moreover, if $w^n \in \mathcal{E}$ for some $n \in \mathbb{N}$, then $w \in \mathcal{E}$.

If $\prod_{n\in\mathbb{N}}(1-\lambda^n) = 0$, then the sequence $(w^n, i^n, \lambda^n)_{n\in\mathbb{N}}$ of Lemma 4.7 naturally defines a sequentially perfect FAP: player i^0 quits first with probability λ^0, and w^1 is the continuation payoff; player i^1 quits with probability λ^1, and w^2 is the continuation payoff, and so on. If $\prod_{n\in\mathbb{N}}(1-\lambda^n) > 0$, then this sequence defines a prefix of an FAP that implements the payoff vector w^0.

Proof. Fix $i \in N$ and let $w \in \overline{\mathcal{E}}_i$. The existence of a sequence $(w^n, i^n, \lambda^n)_{n\in\mathbb{N}}$ satisfying Eq. (11) follows from the fact that for each $M \in \mathbb{I}$ included in $\widehat{N}$, the sets $(\overline{\mathcal{E}}_j)_{j\in M}$ are invariant for $\mathbf{F}_M(\cdot \mid (\overline{\mathcal{E}}_j)_{j\in\widehat{M}})$.

We now prove the second claim. Suppose that there exists $n \in \mathbb{N}$ such that $w^n \in \mathcal{E}$, i.e, there is a sequentially perfect FAP ι with $\gamma^0(\iota) = w^n$. If $n = 0$, then $w = w^0 \in \mathcal{E}$.

If $n \neq 0$, then we know that once the time reaches t^n, a sequentially perfect strategy profile to continue the play and obtain a payoff of w^n is available. We only need to verify that up to time t^n the recursive construction described just before the proof yields a beginning that answers the conditions of sequential perfectness when the continuation payoff at time t^n is w^n. Let $(t^k)_{k=0}^n$ be as follows: $t^0 := 0$ and $t^{k+1} := t^k + (1-t^k)\lambda^k$ for $k = 0, \ldots, n-1$. Define a new FAP ι' by

$$\iota'(t) := \begin{cases} i^k, & \text{if } \ t \in [t^k, t^{k+1}), \ k = 0, \ldots, n-1, \\ \iota\left(\frac{t-t^n}{1-t^n}\right), & \text{if } \ t \in [t^n, 1). \end{cases}$$

By construction, $w = \gamma^0(\iota')$ with $w_i = 0$, $w \in \mathbb{R}^I_+$, and ι' is sequentially perfect. Thus, $w \in \mathcal{E}$. ■

We show now that the sets $(\overline{\mathcal{E}}_i)_{i\in I}$ are supersets of $(\mathcal{E}_i)_{i\in I}$. Our argument is divided into two parts. Lemma 4.8 establishes that $(\overline{\mathcal{E}}_i)_{i\in I}$ is larger than any collection of invariant sets of the Essential APS operator satisfying a certain property. It is then sufficient to show that the collection $(\mathcal{E}_i)_{i\in I}$ satisfies the premise of Lemma 4.8.

Lemma 4.8 *Fix $N \in \mathbb{I}$. Let $(E_i)_{i\in N\cup\widehat{N}}$ be a collection of sets such that for all $M \in \mathbb{I}$ reachable from N including $M = N$,*

- *for each $i \in M$, $E_i \subseteq \mathcal{F}_M$,*
- $(E_i)_{i\in M} = \mathbf{F}_M\left((E_i)_{i\in M} \mid (E_i)_{i\in\widehat{M}}\right)$.

Then, for all $i \in N \cup \widehat{N}$, $E_i \subseteq \overline{\mathcal{E}}_i$.

Proof. We proceed by induction. For $N \in \mathbb{I}$ such that $\widehat{N} = \emptyset$, the result follows directly from Eq. (8) and the fact that $\mathbf{F}_N(\cdot \mid \emptyset)$ is monotone.

Next, consider $N \in \mathbb{I}$ such that $\widehat{N} \neq \emptyset$, and suppose that $E_i \subseteq \overline{\mathcal{E}}_i$ for each $i \in \widehat{N}$. The monotonicity of $\mathbf{T}_N$ in all its arguments implies that

$$(E_i)_{i\in N} = \bigcap_{n=0}^{\infty} \mathbf{F}_N^{(n)}((E_i)_{i\in N} \mid (E_i)_{i\in\widehat{N}}) \subseteq \bigcap_{n=0}^{\infty} \mathbf{F}_N^{(n)}((\mathcal{F}_N)^N \mid (\overline{\mathcal{E}}_i)_{i\in\widehat{N}}) = (\overline{\mathcal{E}}_i)_{i\in N},$$

which completes the inductive step. ■

Proposition 4.9 *$\mathcal{E}_i \subseteq \overline{\mathcal{E}}_i$ for all $i \in I$.*

Proof. We proceed again by induction. In view of Remark 4.6.3 and Lemma 4.8, it is sufficient to prove that $\mathcal{E}_i \subseteq \mathcal{F}_N$ for every strongly connected component N and every $i \in N$. This trivially holds if $\widehat{N} = \emptyset$.

Fix now $N \in \mathbb{I}$ such that $\widehat{N} \neq \emptyset$, and suppose that $(\mathcal{E}_i)_{i\in M} \subseteq (\mathcal{F}_M)^M$ for every $M \subseteq \widehat{N}$. Let $w \in \mathcal{E}_i$ as defined in Lemma 4.5, i.e., $w = \lambda R_i + (1-\lambda)\gamma^0(\iota) \geq \vec{0}$ with $w_i = 0$ for some $(\lambda, \iota) \in [0,1] \times \Upsilon$. We shall show that $w \in \mathcal{F}_N$.

Suppose first that $\iota([0,1)) \subseteq N$. Then, $w \in \mathcal{R}_N$, and the assertion holds trivially.

Suppose next that $t = \inf\{s \geq 0 \ : \ \iota(s) \notin N\} < 1$. By construction, $\gamma^t(\iota) \in \mathcal{E}_{\iota(t)}$, and, by Lemma 4.1, there exists $j \in N$ such that $\gamma_j^t(\iota) = 0$ and $\iota(t) \in S_j$. It follows that w is an element of

$$co\left(\bigcup_{j\in N}\left(\left(\{R_j\} \cup \bigcup_{k\in S_j\cap\widehat{N}} \mathcal{E}_k\right) \cap \mathcal{H}_{\{j\}}\right)\right) \cap \mathbb{R}_+^I.$$

By the induction hypothesis, $(\mathcal{E}_j)_{j\in M} \subseteq (\mathcal{F}_M)^M$ for every $M \subseteq \widehat{N}$, and hence $\mathcal{E}_j \subseteq \overline{\mathcal{E}}_j$ for all $j \in \widehat{N}$. As a result, $w \in \mathcal{F}_N$, and the induction step is complete. ■

The following theorem delivers a condition under which the upper bound on the set of sequentially perfect FAP payoffs is tight, i.e., every payoff vector in $\bigcup_{i\in I} \overline{\mathcal{E}}_i$ can be attained by some sequentially perfect FAP. This condition says that the projection of the set $\mathcal{F}_N$ on the coordinates that corresponds to any set of players in a simple circuit $M \subseteq N$ does not contain the vector $\vec{0}_M$. This implies that for every strongly connected component N, the sets $(\overline{\mathcal{E}}_i)_{i\in N}$ are such that there is no subset of players who can quit consecutively after each other with zero probabilities. This condition is further simplified in two corollaries below and is shown to hold in Example 2.2 and other examples presented in Section 4.4.

Theorem 4.10 *Suppose that for every strongly connected component $N \in \mathbb{I}$, we have $\mathcal{F}_N \cap \mathcal{H}_M = \emptyset$ for all simple circuits $M \subseteq N$. Then,*

$$\mathcal{E} = \bigcup_{i\in I} \overline{\mathcal{E}}_i.$$

Furthermore, for each $w \in \mathcal{E}$, there exists a sequentially perfect FAP ι with $\gamma^0(\iota) = w$ such that the ordinality of $H(\iota)$ is at most the ordinality of $\mathbb{N}$.

Proof. Lemma 4.5 and Proposition 4.9 implies that $\mathcal{E} = \bigcup_{i\in I} \mathcal{E}_i \subseteq \bigcup_{i\in I} \overline{\mathcal{E}}_i$. In what follows, we show the reverse inclusion.

Take $N \in \mathbb{I}$ and suppose that, by induction, we have already shown that for each $i \in \widehat{N}$, (i) the set $\overline{\mathcal{E}}_i$ is compact, (ii) $\overline{\mathcal{E}}_i \subseteq \mathcal{E}$, and (iii) all elements of $\overline{\mathcal{E}}_i$ can be obtained by FAPs ι such that the ordinality of $H(\iota)$ is at most the ordinality of $\mathbb{N}$. Note that the premise is vacuously true whenever $\widehat{N} = \emptyset$.

Property (i) for N is immediate. Indeed, since $\mathbf{F}_N(\cdot \mid (\overline{\mathcal{E}}_i)_{i\in\widehat{N}})$ maps compact sets to compact sets and $\mathcal{F}_N$ is compact, the sets $(\overline{\mathcal{E}}_i)$ are compact as the intersection of compact sets.

We now establish properties (ii) and (iii) for N. Fix $i \in N$ and $w \in \overline{\mathcal{E}}_i$. Suppose first that there exists a sequence $(w^n, i^n, \lambda^n)_{n\in\mathbb{N}}$ satisfying Eq. (11) such that $\lambda^n = 1$ or $i^{n+1} \in \widehat{N}$ for some $n \in \mathbb{N}$. In the former case, we must have $w^n = R_{i^n} \geq \vec{0}$, while in the latter case, $w^{n+1} \in \overline{\mathcal{E}}_{i^{n+1}} \subseteq \mathcal{E}$. In either case, Lemma 4.7 implies $w \in \mathcal{E}$.

Suppose now that any sequence $(w^n, i^n, \lambda^n)_{n\in\mathbb{N}}$ satisfying Eq. (11) is such that $\lambda^n < 1$ and $i^{n+1} \in N$ for all $n \in \mathbb{N}$. Define a sequence $(t^n)_{n\in\mathbb{N}}$ by setting $t^0 := 0$ and $t^{n+1} := t^n + (1-t^n)\lambda^n$ for all $n \in \mathbb{N}$. By construction, $(t^n)_{n\in\mathbb{N}}$ is nondecreasing and bounded from above by 1. We claim that this sequence converges to 1. To establish this claim, consider the following infimum:

$$
\begin{aligned}
\nu := \quad & \inf_{\left((\widetilde{i}^k)_{k=1}^{|N|}, \widetilde{w}, (\widetilde{\lambda}^k)_{k=1}^{|N|-1}\right) \in N^{|N|}\times\mathcal{F}_N\times[0,1]^{|N|-1}} \quad \textstyle\sum_{k=1}^{|N|-1} \widetilde{\lambda}^k, \\
& \text{s.t.} \quad \widetilde{i}^{k+1} \in S_{\widetilde{i}^k} \ \text{ for } \ k = 1, \ldots, |N|-1, \ \text{ and} \\
& \begin{cases}
\widetilde{w}_{\widetilde{i}^1} = 0, \\
\widetilde{w}_{\widetilde{i}^2} = \widetilde{\lambda}^1 R_{\widetilde{i}^1, \widetilde{i}^2}, \\
\widetilde{w}_{\widetilde{i}^k} = \widetilde{\lambda}^1 R_{\widetilde{i}^1, \widetilde{i}^k} + \prod_{m=1}^{k-2}(1-\widetilde{\lambda}^m)\widetilde{\lambda}^{k-1} R_{\widetilde{i}^{k-1}, \widetilde{i}^m} \text{ for } k = 3, ..., |N|.
\end{cases}
\end{aligned}
\tag{12}
$$

The quantity $\sum_{k=1}^{|N|-1} \widetilde{\lambda}^k$ serves as a proxy to the total probability of quitting of $|N|-1$ consecutive players in the full sequence $(w^n, i^n, \lambda^n)_{n\in\mathbb{N}}$ satisfying Eq. (11). Thus, ν is the infimum of these quantities.

We will show that the condition in Theorem 4.10 implies $\nu > 0$, which will further imply that the sequence $(t^n)_{n\in\mathbb{N}}$ converges to 1. By way of contradiction, assume that $\nu = 0$. Since $\mathcal{F}_N$ is compact, the infimum is attained by some point, say $\left((\widetilde{i}^k)_{k=1}^{|N|}, \widetilde{w}, (\widetilde{\lambda}^k)_{k=1}^{|N|-1}\right)$. By assumption, $\nu = 0$ which implies that $\widetilde{w}_j = 0$ for every $j \in \{\widetilde{i}^k\}_{k=1}^{|N|}$. Since $\widetilde{i}^{k+1} \in S_{\widetilde{i}^k}$ for $k = 1, \ldots, |N|-1$, there exists a simple circuit M in $\{\widetilde{i}^k\}_{k=1}^{|N|}$. The existence of a simple circuit and a payoff in which all players in that simple circuit get zero contradict the assumption of the theorem. It follows that $\nu > 0$.

The reader can verify that, for each $n \in \mathbb{N}$, the point $\left((i^{n+k})_{k=1}^{|N|}, w^n, (\lambda^{n+k})_{k=1}^{|N|-1}\right)$ satisfies the constraints of the auxiliary problem (12). As a result, $\sum_{k=1}^{|N|-1} \lambda^{n+k} \geqslant \nu > 0$ for every $n \in \mathbb{N}$.

Going back to the sequence $(t^n)_{n\in\mathbb{N}}$, note that $t^{n+|N|-1} - t^n = \sum_{k=1}^{|N|-1} \lambda^{n+k}(1 - t^{n+k-1})$ for every $n \in \mathbb{N}$. Since this latter sequence is non-decreasing, for each $n \in \mathbb{N}$, it has to satisfy the following:

$$t^{n+|N|-1} - t^n \geq \sum_{k=1}^{|N|-1} \lambda^{n+k}(1 - t^{n+|N|-1}) \geq \nu(1 - t^{n+|N|-1}).$$

It follows that $(t^n)_{n\in\mathbb{N}}$ converges to 1 as n tends to ∞. Since $(t^n)_{n\in\mathbb{N}}$ converges to 1, the following ι constitutes an FAP:

$$\iota(t) := i^n \text{ whenever } t \in [t^n, t^{n+1}).$$

By construction, ι is sequentially perfect with $w = \gamma^0(\iota)$ and $w \in \mathcal{E}$.

Finally, note that in each of the two cases, the ordinality of the constructed FAP ι is at most the ordinality of $\mathbb{N}$. This completes the induction step. ■

Theorem 4.10 provides the condition under which the Essential APS operator can be used to characterize the set of all payoff vectors attainable by sequentially perfect FAPs. We note that this condition is based on the collection $(\mathcal{F}_N)_{N\in\mathbb{I}}$, and thus can only be checked while running the algorithm if there are several strongly connected components. Corollary 4.11 presents a simpler condition replacing $(\mathcal{F}_N)_{N\in\mathbb{I}}$ with an alternative collection $(\overline{\mathcal{F}}_N)_{N\in\mathbb{I}}$ such that $\overline{\mathcal{F}}_N \supseteq \mathcal{F}_N$ for each $N \in \mathbb{I}$. These sets $(\overline{\mathcal{F}}_N)_{N\in\mathbb{I}}$ can be computed before running the algorithm.

Specifically, for every $N \in \mathbb{I}$ such that $\widehat{N} = \emptyset$, set $\overline{\mathcal{F}}_N := \mathcal{R}_N$, and, otherwise, set

$$\overline{\mathcal{F}}_N := co\left(\bigcup_{i\in N}\left(\left(\{R_i\} \cup \bigcup_{M\in\mathbb{I}:S_i\cap M\neq\emptyset} \mathcal{R}_{M\cup\widehat{M}}\right) \cap \mathcal{H}_{\{i\}}\right)\right) \cap \mathbb{R}^I_+. \tag{13}$$

The definition of $\overline{\mathcal{F}}_N$ in (13) resembles (9), replacing $\overline{\mathcal{E}}_j$ with a larger set, namely $\mathcal{R}_{M\cup\widehat{M}}$, where M is a strongly connected component containing player j. By construction, $\overline{\mathcal{F}}_N$ is a superset of $\mathcal{F}_N$, and we obtain at once.

Corollary 4.11 *Suppose that for every strongly connected component $N \in \mathbb{I}$, we have $\overline{\mathcal{F}}_N \cap \mathcal{H}_M = \emptyset$ for all simple circuits $M \subseteq N$. Then,*

$$\mathcal{E} = \bigcup_{i\in I} \overline{\mathcal{E}}_i.$$

If there is only one strongly connected component, then the conditions of Theorem 4.10 and its corollary coincide. In this case, the Essential APS operator is particularly tractable and can be explicitly unpacked as documented in the corollary. It holds in particular in Example 2.2.

Corollary 4.12 *Suppose that for each $i \in I$, $S_i = \{i+1\}$, where the addition is modulo $|I|$. Then $\mathbb{I} = \{I\}$ and, if $\vec{0} \notin \mathcal{R}_I$, then for each $i \in I$,*

$$\overline{\mathcal{E}}_i = \bigcap_{n=0}^{\infty} \left(\mathbf{T}_i \circ \mathbf{T}_{i+1} \circ \ldots \circ \mathbf{T}_{i+|I|}\right)^{(n)}(\mathcal{R}_I) \ \ \textit{and} \ \ \mathcal{E} = \bigcup_{i \in I} \overline{\mathcal{E}}_i.$$

Finally, we note that if the condition of Theorem 4.10 fails, then the upper bounds $(\overline{\mathcal{E}}_i)_{i \in I}$ might not be tight. We now provide an example, which illustrates that the assumption of Theorem 4.10 cannot be easily dispensed with.

Example 4.13 *Consider the quitting game with five players and the following payoff matrix of single quittings:*

$$R = \begin{pmatrix} 0 & 2 & -\frac{1}{2} & 1 & -1 \\ -\frac{1}{2} & 0 & 2 & 1 & -1 \\ 2 & -\frac{1}{2} & 0 & 1 & -1 \\ -1 & -2 & -3 & 0 & \frac{10}{7} \\ 2 & \frac{7}{2} & \frac{47}{8} & -\frac{5}{12} & 0 \end{pmatrix}.$$

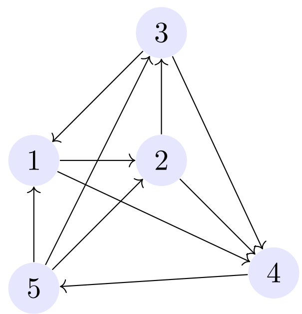


Figure 2: The directed graph (I, L) in Example 4.13.

In this game, $S_1 = \{2,4\}$, $S_2 = \{3,4\}$, $S_3 = \{1,4\}$, $S_4 = \{5\}$, and $S_5 = \{1,2,3\}$; see Figure 2 for an illustration of the graph (I, L). The reader can verify that I is strongly connected and therefore the only strongly connected component of $\mathbb{I}$.

We claim $\bigcup_{i \in I} \overline{\mathcal{E}}_i \neq \mathcal{E}$. To this end, note that there exists a point $w \in \mathcal{R}_I$ such that $w_1 = w_2 = w_3 = 0$, $w_4 > 0$, and $w_5 > 0$, i.e.,

$$w = \left(0,0,0,\frac{20}{71},\frac{20}{71}\right) = \frac{5}{71} \cdot R_1 + \frac{686}{2911} \cdot R_2 + \frac{684}{2911} \cdot R_3 + \frac{1309}{2911} \cdot R_4 + \frac{552}{2911} \cdot R_5.$$

On the one hand, this point cannot be eliminated during iterations of the essential APS operator, because $w \in \mathbf{T}_i(\{w\})$ for $i = 1,2,3$ and $\{1,2,3\}$ is a simple circuit. As a result, $w \in \overline{\mathcal{E}}_i$ for $i = 1,2,3$.

On the other hand, w cannot be attained by any sequentially perfect FAP. Indeed, since $R_{i,i+1} > 0$ for $i = 1,2,3$, where the addition is modulo 3, there is no $\lambda \in (0,1]$ and $v \in \mathcal{R}_I \subseteq \mathbb{R}^I_+$ such that $w = \lambda R_i + (1-\lambda)v$. It follows that $w \notin \mathcal{E}_1 \cup \mathcal{E}_2 \cup \mathcal{E}_3$. At the same *time, $w \notin \mathcal{E}_4 \cup \mathcal{E}_5$, because $w_4 > 0$ and $w_5 > 0$. We conclude that there is no sequentially perfect FAP ι such that $\gamma_0(\iota) = w$. We shall see in the next section, that, for this example, there are sequentially perfect FAP, which do not fit to our construction, because they have accumulation points different from 1.*

4.4 Examples: Numerical Results and Explicit Constructions

In this section, we illustrate how Lemma 4.1 can be used to qualitatively characterize the set of sequentially perfect FAPs. We also show the Essential APS approach in action and numerically compute the set $\mathcal{E}$ in a series of examples. We provide four examples. First, we show how Lemma 4.1 can be used to formally establish that the set of sequentially perfect FAPs is empty in Example 3.4 and then confirm this conclusion numerically. Second, we provide two computed examples that explain how our approach can be used to infer the structure of sequentially perfect FAPs and construct them systematically. Finally, we illustrate that our approach can be useful to compute some sequentially perfect FAPs even when the condition of Theorem 4.10 is not satisfied.

Our numerical implementation of the Essential APS approach is done in Julia using the "Polyhedra" package and arbitrary precision arithmetic, i.e., all variables were stored as instances of the "BigInt" type. A basic computational step takes a convex compact set $E \subset \mathbb{R}^I_+$ with a finite number of extreme points and some player i. Then, it applies $\mathbf{T}_i$ to E to produce another convex compact set

$$\mathbf{T}_i(E) = co\left(\{R_i\} \cup E\right) \cap \mathcal{H}_{\{i\}} \cap \mathbb{R}^I_+$$

with a finite number of extreme points. This construction naturally extends to sets that can be written as a finite union of convex sets.

The numerical implementation terminates after a few iterations in the first two examples and yields finer and finer approximations as the number of iterations increases in the last two examples. To improve readability, we report the collection $(\overline{\mathcal{E}}_i)_{i \in N}$ using floating-point arithmetic with an integer exponent of base 3 when a rational representation is too cumbersome.

Example 3.4 continued. *Recall that in this example there are four players with the following payoff matrix of single quittings:*

$$R = \begin{pmatrix} 0 & 4 & -1 & -1 \\ -1 & 0 & 4 & 1 \\ -1 & -1 & 0 & 4 \\ 4 & 1 & -1 & 0 \end{pmatrix}.$$

The reader can verify that $S_i = \{i+1\}$ for every $i \in I$, where the addition is modulo 4, and the graph (I, L) is a simple circuit. Therefore, by Lemma 4.1, the players must quit in a cyclic order in any sequentially perfect FAP.

We now use the topology of (I, L) to show that there cannot exist a sequentially perfect FAP. To this end, let ι be sequentially perfect. Fix $t \in (0,1)$ and inductively define $(t^n)_{n \in \mathbb{N}}$

as follows: $t^0 := t$ *and* $t^{n+1} := \inf\{s \geq t^n : \iota(s) \neq \iota(t^n)\} \in (t^n, 1)$. *If* $i = \iota(t^n)$, *then the payoffs at two consecutive time instances,* t^n *and* t^{n+1}, *satisfy* $\gamma^{t^n}(\iota) \in \mathbf{T}_i(\{\gamma^{t^{n+1}}(\iota)\})$; *specifically, by Eq.* (1),

$$\gamma^{t^n}(\iota) = \frac{t^{n+1} - t^n}{1 - t^n} R_i + \frac{1 - t^{n+1}}{1 - t^n} \gamma^{t^{n+1}}(\iota).$$

Since the next player who quits at t^{n+1} *is necessarily* $i + 1$, *we have* $\gamma_{i+1}^{t^{n+1}}(\iota) = 0$, *which gives* $\frac{t^{n+1} - t^n}{1 - t^n} = \frac{\gamma_i^{t^n}(\iota)}{R_{i,i+1}}$.

Substitute the expression for player i*'s quitting probability into the above relation to solve for* $\gamma^{t^{n+1}}(\iota)$ *as a function of* $\gamma^{t^n}(\iota)$. *By Lemma 4.1, since player* i *is preceded by player* $i+3$, *we necessarily have* $\gamma_{i+3}^{t^n}(\iota) = 0$. *The reader can verify that*

$$\frac{\gamma_{i+3}^{t^{n+1}}(\iota)}{\gamma_{i+2}^{t^{n+1}}(\iota)} = \frac{1}{4\frac{\gamma_{i+2}^{t^n}(\iota)}{\gamma_{i+1}^{t^n}(\iota)} - R_{i,i+2}}, \tag{14}$$

where $R_{i,i+2} = -1$ *for* $i = 1, 3$ *and* $R_{i,i+2} = 1$ *for* $i = 2, 4$.

Toward a contradiction, for every $n \in \mathbb{N}$, *set* $z^n := \frac{\gamma_{\iota(t^n)+2}^{t^n}(\iota)}{\gamma_{\iota(t^n)+1}^{t^n}(\iota)}$. *By Eq.* (14), *if* $\iota(t) = i$, *then*

$$z^{n+2} = \frac{1}{4\left(\frac{1}{4z^n - 1}\right) + 1} = \frac{4z^n - 1}{4z^n + 3} \quad \forall n \in \mathbb{N}.$$

Starting from any $z^0 \geq 0$, *there exists* $n \in \mathbb{N}$ *such that* $z^n < 0$, *which is impossible. It follows that* ι *is not a sequentially perfect FAP.*

We now illustrate how the same conclusion can be obtained numerically using the Essential APS approach. Note that the game satisfies the condition of Theorem 4.10. Since the graph is a simple circuit, by Corollary 4.12, the sets $(\overline{\mathcal{E}}_i)_{i \in I}$ *can be re-written as follows: for each* $i \in I$,

$$\overline{\mathcal{E}}_i = \mathbf{T}_i(\overline{\mathcal{E}}_{i+1}) = (\mathbf{T}_i \circ \mathbf{T}_{i+1} \circ \mathbf{T}_{i+2} \circ \mathbf{T}_{i+3})(\overline{\mathcal{E}}_i).$$

To show this, we proceed backwards along the simple circuit applying repeatedly $\mathbf{T}_i$*'s:*

- *As mentioned on Page 13, the set* $\mathcal{R}_I \cap \{w \geq \vec{0} : w_1 = 0\}$ *is the convex hull of* $(0, \frac{5}{2}, 0, 0)$, $(0, 0, 0, \frac{5}{2})$, $(0, 0, \frac{11}{5}, \frac{7}{5})$, $(0, \frac{17}{9}, \frac{11}{9}, 0)$, $(\frac{11}{5}, \frac{7}{5}, 0, 0)$, $(0, \frac{1}{5}, 3, \frac{4}{5})$.
- *The set* $(\mathbf{T}_1 \circ \mathbf{T}_2 \circ \mathbf{T}_3 \circ \mathbf{T}_4)(\mathcal{R}_I)$ *is the convex hull of* $(0, 0, 0, \frac{5}{2})$, $(0, 0, 1, 2)$, $(0, 2, 0, 1)$.
- *The set* $(\mathbf{T}_1 \circ \mathbf{T}_2 \circ \mathbf{T}_3 \circ \mathbf{T}_4)^2(\mathcal{R}_I)$ *is empty.*

It follows that $\mathcal{E} = \emptyset$.

We next provide an example in which the set $\mathcal{E} = \bigcup_{i \in I} \overline{\mathcal{E}}_i$ is nonempty and is strictly smaller than $\mathcal{R}_I$. In other words, the set of attainable payoffs cannot be computed by the naive APS approach, but the Essential APS approach does the job.

Example 4.14 *There are five players with the following payoff matrix of single quittings:*

$$R = \begin{pmatrix} 0 & 4 & -\frac{1}{2} & -1 & -1 \\ -1 & 0 & 3 & -1 & \frac{1}{2} \\ -\frac{1}{8} & -1 & 0 & 4 & -1 \\ -1 & -\frac{1}{2} & -1 & 0 & 4 \\ 5 & 1 & -1 & -1 & 0 \end{pmatrix}.$$

In this game, $S_i = \{i+1\}$ for each $i \in I$, where addition is modulo 5, and the graph (I, L) is a simple circuit. According to Corollary 4.12, the sets $(\overline{\mathcal{E}}_i)_{i\in I}$ can be re-written as follows: for each $i \in I$,

$$\overline{\mathcal{E}}_i = \mathbf{T}_i(\overline{\mathcal{E}}_{i+1}) = (\mathbf{T}_i \circ \mathbf{T}_{i+1} \circ \mathbf{T}_{i+2} \circ \mathbf{T}_{i+3} \circ \mathbf{T}_{i+4})(\overline{\mathcal{E}}_i).$$

To compute the collection $(\overline{\mathcal{E}}_i)_{i\in I}$, we consecutively apply the operators $(\mathbf{T}_i)_{i\in I}$ to $\mathcal{R}_I$. Our implementation of the algorithm produces an exact solution in a few iterations. However, since $(\overline{\mathcal{E}}_i)_{i\in I}$ consists of cumbersome rationals, we report their approximations using floating-point arithmetic. Each set in the collection $(\overline{\mathcal{E}}_i)_{i\in I}$ is convex, and thus can be described by its extreme points. Figure 3 depicts these extreme points: $\overline{\mathcal{E}}_1$ has 7 extreme points, $\overline{\mathcal{E}}_2$ has 4 extreme points, and $\overline{\mathcal{E}}_3$, $\overline{\mathcal{E}}_4$, and $\overline{\mathcal{E}}_5$ have 6 extreme points each.

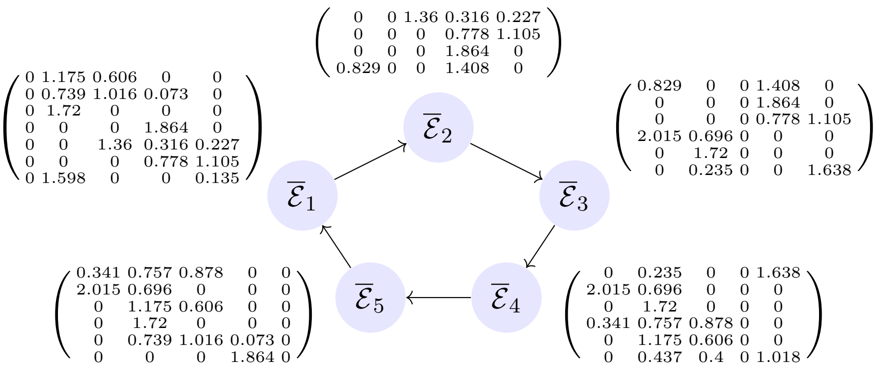


Figure 3: Collection $(\overline{\mathcal{E}}_i)_{i\in I}$ in Example 4.14.
Next to each set $\overline{\mathcal{E}}_i$ appear its extreme points.

Since the game satisfies the condition of Theorem 4.10 and Corollary 4.12, the union of sets in $(\overline{\mathcal{E}}_i)_{i\in I}$ coincides with the whole set of payoffs that can be attained by sequentially perfect FAPs.

The representation depicted in Figure 3 is quite informative about the structure of sequentially perfect FAPs. For example, let ι be sequentially perfect, and consider $t \in (0,1)$ with $\iota(t) = 1$. The previous player to quit must be Player 5, therefore $\gamma^t(\iota)$ is necessarily a convex combination of only the first 4 extreme points in $\overline{\mathcal{E}}_1$. Since Player 1 should quit at a positive rate, Player 2's payoff at t must be positive, thus this convex combination necessarily puts positive weights on the first three extreme points in $\overline{\mathcal{E}}_1$. The same logic applies to other players. As a result, if we knew how to generate extreme points in which a preceding *player obtains* 0 *and a successive player obtains a strictly positive payoff, we would be able to build all sequentially perfect FAPs.*

To see how to generate those extreme points, consider, for example, the first extreme point in $\overline{\mathcal{E}}_1$*, that is,* $w = (0, 1.175, 0.606, 0, 0)$*. If* $w = \gamma^t(\iota)$*, then Player* 1*'s total quitting probability* λ *satisfies* $1.175 = R_{1,2}\lambda = 4\lambda$*, because both Players* 1 *and* 2*'s continuation payoffs must be* 0*. This gives the continuation vector* $v = (0, 0, 1.066, 0.416, 0.416) \in \overline{\mathcal{E}}_2$*, i.e., the point such that* $w \in \mathbf{T}_i(\{v\})$*.*

Proceeding inductively in this way, we can then construct an automaton over the extreme points of $(\overline{\mathcal{E}}_i)_{i\in I}$ *that characterizes players' quittings rates in every sequentially perfect FAP.*

In both examples above, there is only one strongly connected component, which forms a simple circuit. As a result, for each player $i \in I$, the upper bound $\overline{\mathcal{E}}_i$ is convex. As the following example illustrates, this is not necessarily the case when there are multiple strongly connected components.

Example 4.15 *Consider the game with six players and the following payoff matrix of single quittings:*

$$R = \begin{pmatrix} 0 & 2 & -\frac{1}{2} & 1 & 1 & 1 \\ -\frac{1}{2} & 0 & 2 & 1 & 1 & 1 \\ 2 & -\frac{1}{2} & 0 & 1 & 1 & 1 \\ -\frac{3}{8} & 1 & 1 & 0 & 2 & -\frac{1}{2} \\ 1 & 2 & 1 & -\frac{1}{2} & 0 & 2 \\ 2 & 1 & 1 & 2 & -\frac{1}{2} & 0 \end{pmatrix}.$$

Figure 4 depicts the graph (I, L)*. There are two strongly connected components* $\mathbb{I} = \{\{1, 2, 3\}, \{4, 5, 6\}\}$*, and* $\{4, 5, 6\}$ *is reachable from* $\{1, 2, 3\}$*.*

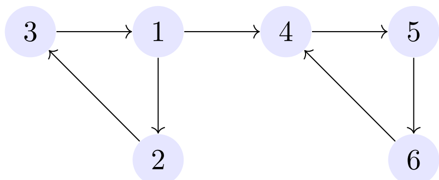


Figure 4: The directed graph (I, L) in Example 4.15.

We now use the Essential APS approach to compute the sets $(\overline{\mathcal{E}}_i)_{i\in I}$*. First, consider the strongly connected component* $\{4, 5, 6\}$*. Similarly to Example 2.2, the sets of payoffs attainable by sequentially perfect FAPs in which only players* 4*,* 5*, and* 6 *quit coincide with the boundary of a certain triangle:* $\overline{\mathcal{E}}_4 = \mathcal{R}_{\{4,5,6\}} \cap \mathcal{H}_{\{4\}}$ *is the convex hull of* $(0, \frac{25}{21}, 1, 0, \frac{3}{2}, 0)$ *and* $(\frac{9}{8}, \frac{37}{21}, 1, 0, 0, \frac{3}{2})$*,* $\overline{\mathcal{E}}_5 = \mathcal{R}_{\{4,5,6\}} \cap \mathcal{H}_{\{5\}}$ *is the convex hull of* $(\frac{9}{8}, \frac{37}{21}, 1, 0, 0, \frac{3}{2})$ *and* $(\frac{3}{2}, \frac{22}{21}, 1, \frac{3}{2}, 0, 0)$*, and* $\overline{\mathcal{E}}_6 = \mathcal{R}_{\{4,5,6\}} \cap \cap \mathcal{H}_{\{6\}}$ *is the convex hull of* $(\frac{3}{2}, \frac{22}{21}, 1, \frac{3}{2}, 0, 0)$ *and* $(0, \frac{25}{21}, 1, 0, \frac{3}{2}, 0)$*. In what follows, we analytically solve for the remaining collection* $(\overline{\mathcal{E}}_i)_{i\in\{1,2,3\}}$*, and then compare the exact solution to the numerical approximation of this set.*

We focus on $\overline{\mathcal{E}}_1$ because the other two sets can be derived from it by $\overline{\mathcal{E}}_3 = \mathbf{T}_3(\overline{\mathcal{E}}_1)$ and $\overline{\mathcal{E}}_2 = \mathbf{T}_2(\overline{\mathcal{E}}_3)$. Iterate along the simple circuit $\{1, 2, 3\}$ to obtain the following:

$$\begin{aligned}\overline{\mathcal{E}}_1 = \mathbf{T}_1(\overline{\mathcal{E}}_2) \cup \mathbf{T}_1(\overline{\mathcal{E}}_4) &= (\mathbf{T}_1 \circ \mathbf{T}_2 \circ \mathbf{T}_3)(\overline{\mathcal{E}}_1) \cup \mathbf{T}_1(\overline{\mathcal{E}}_4) = \\ &= (\mathbf{T}_1 \circ \mathbf{T}_2 \circ \mathbf{T}_3)^{(n+1)}(\overline{\mathcal{E}}_1) \cup \bigcup_{k=0}^{n} (\mathbf{T}_1 \circ \mathbf{T}_2 \circ \mathbf{T}_3)^{(k)}(\mathbf{T}_1(\overline{\mathcal{E}}_4)), \quad (15)\end{aligned}$$

which holds for every $n \in \mathbb{N}$.

We first need to make two preliminary calculations. The set $\mathbf{T}_1(\overline{\mathcal{E}}_4)$ coincides with $\mathbf{T}_1(\{v^0\})$ for $v^0 := (0, \frac{25}{21}, 1, 0, \frac{3}{2}, 0)$, because v^0 is the only point in $\overline{\mathcal{E}}_4$ that gives 0 to Player 1. Since $R_{1,3}$ is the only negative entry in R_1, $\mathbf{T}_1(\{v^0\})$ is the convex hull of v^0 and another extreme point w^0 in which Player 3 obtains 0. By definition, $w^0 = \lambda R_1 + (1-\lambda)v^0$ for some $\lambda \in [0, 1]$ and $w^0_2 = 0$. Therefore, $\lambda = \frac{2}{3}$, and hence $w^0 = (0, \frac{109}{63}, 0, \frac{2}{3}, \frac{7}{6}, \frac{2}{3})$.

Second, let v be a non-negative continuation payoff that gives 0 to players 1 and 3. We shall compute $(\mathbf{T}_1 \circ \mathbf{T}_2 \circ \mathbf{T}_3)(\{v\})$. Take some point w in this set. By definition, $w = \lambda R_1 + (1-\lambda)w'$, $w' = \lambda' R_2 + (1-\lambda')w''$, $w'' = \lambda'' R_3 + (1-\lambda'')v$ for some $(\lambda, \lambda', \lambda'') \in [0, 1]^3$ and $w'_1 = w'_2 = w''_2 = w''_3 = 0$. Since only $R_{i,i+2} < 0$ for $i = 1, 2, 3$, proceeding backwards, we obtain $\lambda'' = \frac{2v_2}{1+2v_2}$, $\lambda' = \frac{2w''_1}{1+2w''_1}$, and λ is any number in the interval $\left[\frac{2w'_3}{1+2w'_3}, 1\right]$. It follows that $(\mathbf{T}_1 \circ \mathbf{T}_2 \circ \mathbf{T}_3)(\{v\})$ is the set of convex combinations of just two points, $\phi(v)$ and $\psi(v)$, defined by

$$\begin{aligned}\phi(v) &:= \left(0, 0, \frac{16v_2}{10v_2+1}, \frac{10v_2+v_4}{10v_2+1}, \frac{10v_2+v_5}{10v_2+1}, \frac{10v_2+v_6}{10v_2+1}\right), \\ \psi(v) &:= \left(0, \frac{64v_2}{42v_2+1}, 0, \frac{42v_2+v_4}{42v_2+1}, \frac{42v_2+v_5}{42v_2+1}, \frac{42v_2+v_6}{42v_2+1}\right).\end{aligned}$$

Since only $\psi(v)$ gives the third player the payoff of 0, for each $n \in \mathbb{N}$ the set $(\mathbf{T}_1 \circ \mathbf{T}_2 \circ \mathbf{T}_3)^{(n+1)}(\{v\})$ coincides with convex combinations of $\phi(\psi^{(n)}(v))$ and $\psi^{(n+1)}(v)$.

We now have all ingredients that are necessary to analytically compute the set $\overline{\mathcal{E}}_1$. The reader can verify that $\psi^{(n+1)}(v)$ converges to $(0, \frac{3}{2}, 0, 1, 1, 1)$ irrespective of v. Therefore, letting $n \to \infty$ in Eq. (15), we obtain that $\overline{\mathcal{E}}_1$ is given by the following union:

$$\overline{\mathcal{E}}_1 = \left\{(0, \frac{3}{2}, 0, 1, 1, 1)\right\} \cup \bigcup_{n=0}^{\infty} co(\{v^n, w^n\}),$$

where (v^0, w^0) are as defined above, and $(v^{n+1}, w^{n+1}) := (\phi(w^n), \psi(w^n))$ for all $n \in \mathbb{N}$. To help visualize this set, in Table 1 we compute the first three elements corresponding to $n = 0, 1, 2$ using floating-point arithmetic with an exponent of base 3.

We end this example with a short discussion on our numerical implementation. The set $\mathcal{F}_{\{1,2,3\}} \cap \{w \in \mathbb{R}^I_+ | w_1 = 0\}$ is the convex hull of $(0, \frac{109}{63}, 0, \frac{2}{3}, \frac{7}{6}, \frac{2}{3})$, $(0, \frac{3}{2}, 0, 1, 1, 1)$, $(0, 0, \frac{3}{2}, 1, 1, 1)$, $(0, 0, \frac{421}{271}, \frac{250}{271}, \frac{563}{542}, \frac{250}{271})$, and $(0, \frac{25}{21}, 1, 0, \frac{3}{2}, 0)$. After $n+1$ iterations, the Essential APS algorithm outputs the following:

$$(\mathbf{T}_1 \circ \mathbf{T}_2 \circ \mathbf{T}_3)^{(n+1)}(\mathcal{F}_{\{1,2,3\}}) \cup \bigcup_{k=0}^{n} (\mathbf{T}_1 \circ \mathbf{T}_2 \circ \mathbf{T}_3)^{(k)}(\mathbf{T}_1(\overline{\mathcal{E}}_4)).$$

	v^n	w^n
$n=0$	$(0, 1.19, 1, 0, 1.5, 0)$	(0,1.73,0,0.667,1.167,0.667)
$n=1$	$(0, 0, 0.965, 0.982, 1.009, 0.982)$	$(0, 1.503, 0, 0.995, 1.002, 0.995)$
$n=2$	$(0, 0, 0.960, 1, 1, 1)$	$(0, 1.5, 0, 1, 1, 1)$

Table 1: Representation of $\{(v^n, w^n)\}_{n=0,1,2}$ using float-point arithmetic in Example 4.15.

As discussed above, the first term converges to $(0, \frac{3}{2}, 0, 1, 1, 1)$ and the second term corresponds to $\{(v^k, w^k)\}_{k=0}^n$. So, the Essential APS approach provides a finer and finer approximation of $\overline{\mathcal{E}}_1$ as the number of iterations increases. Moreover, since the assumptions of Theorem 4.10 and Corollary 4.11 are satisfied, $\mathcal{E}$ coincides with the union of $(\overline{\mathcal{E}}_i)_{i\in I}$.

Our approach is valid when the ordinality of $H(\iota)$ is at most the ordinality of $\mathbb{N}$. We will now revisit Example 4.13 to show that even when this condition does not hold, the Essential APS approach can be used to characterize some subsets of $\mathcal{E}$.

Example 4.13 continued. *Recall that in this example there are five players and the payoff matrix of single quittings is*

$$R = \begin{pmatrix} 0 & 2 & -\frac{1}{2} & 1 & -1 \\ -\frac{1}{2} & 0 & 2 & 1 & -1 \\ 2 & -\frac{1}{2} & 0 & 1 & -1 \\ -1 & -2 & -3 & 0 & \frac{10}{7} \\ 2 & \frac{7}{2} & \frac{47}{8} & -\frac{5}{12} & 0 \end{pmatrix}.$$

As pointed out on page 23, the Essential APS approach only provides an upper bound to the set $\mathcal{E}$. We will show that there is a sequentially perfect FAP ι with an infinite cycle, which should be contrasted with the claim of Theorem 4.10. Then, we will illustrate how the payoff along this FAP can be numerically approximated using our technique.

The set $H(\iota)$ will be order equivalent to $\mathbb{N}\times\mathbb{N}$, specifically, denoting by ω the first infinite ordinal, we will have $H(\iota) = \{t^{k\omega+n} : k, n \in \mathbb{N}\}$ for a certain sequence $(t^{k\omega+n})_{k,n\in\mathbb{N}}$. To formally define this FAP, we need the following auxiliary sequence $(i^n, \lambda^n)_{n\in\mathbb{N}}$ given by

n	0	1	2	3	4	5	6	7	$\cdots$	n	$\cdots$
i^n	4	5	1	2	3	1	2	3	$\cdots$	$(n-2) \mod 3$	$\cdots$
λ^n	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{4}$							$\frac{3}{4+14\cdot 4^{n-3}}$	$\cdots$

Define recursively $(t^{k\omega+n})_{k,n\in\mathbb{N}}$ as follows:

$$\begin{aligned} t^{kw} &:= 1 - \left(\frac{5}{12}\right)^k, \quad \forall k \in \mathbb{N}, \\ t^{k\omega+n+1} &:= t^{k\omega+n} + (1 - t^{k\omega+n})\lambda^n, \quad \forall k, n \in \mathbb{N}, \end{aligned}$$

and set the FAP ι to be

$$\iota(t) := i^{k\omega+n}, \qquad \text{whenever } t \in [t^{k\omega+n}, t^{k\omega+n+1}).$$

We note that $t^{k\omega+n}$ *converges to* 1 *as* $k, n \to \infty$. *Thus, the function* ι *is, in a sense, periodic with period* ∞. *It is not difficult, though a bit tedious, to verify that this FAP is sequentially perfect, i.e.,* $\gamma^{t^{k\omega+n}}(\iota) \geq \vec{0}$ *and* $\gamma^{t^{k\omega+n}}_{i^{k\omega+n}}(\iota) = 0$ *for every* $k, n \in \mathbb{N}$. *In fact,*

$$\begin{array}{rcl}\gamma^{0}(\iota) & = & (0,0,0,0,\frac{5}{7}) \\ \gamma^{t^1}(\iota) & = & (1,2,3,0,0), \\ \gamma^{t^2}(\iota) & = & (0,\frac{1}{2},\frac{1}{8},\frac{5}{12},0), \\ \gamma^{t^n}(\iota) & \to & \gamma^{0}(\iota), \\ \gamma^{t^{k\omega+n}}(\iota) & = & \gamma^{t^{k'\omega+n}}(\iota), \quad \forall k,k',n \in \mathbb{N}.\end{array}$$

We now numerically approximate the payoffs of this orbit using our approach. Specifically, for some fixed $k \geq 1$, *we solve for the set of payoff vectors which are attainable by sequentially perfect FAPs in which players* 1, 2, *and* 3 *cycle for* k *times and then players* 4 *and* 5 *quit consecutively. Let* $\mathcal{U}^k$ *be the set of such equilibrium payoffs at the beginning of the cycle, when player* 1 *is the quitter. Then,*

$$\mathcal{U}^k \subseteq ((\mathbf{T}_1 \circ \mathbf{T}_2 \circ \mathbf{T}_3)^{(k)} \circ \mathbf{T}_4 \circ \mathbf{T}_5)(\mathcal{U}^k).$$

As before, we calculate an upper bound $\overline{\mathcal{U}^k}$ *of* $\mathcal{U}^k$ *using iterations, starting with the set of feasible and rational payoffs:*

$$\overline{\mathcal{U}}^k := \bigcap_{n=0}^{\infty} \Big((\mathbf{T}_1 \circ \mathbf{T}_2 \circ \mathbf{T}_3)^{(k)} \circ \mathbf{T}_4 \circ \mathbf{T}_5\Big)^{(n)} (\mathcal{R}_I).$$

Since $\vec{0} \notin \overline{\mathcal{U}}^k$, *by the same argument as the one used in the proof of Theorem 4.10, every point in* $\overline{\mathcal{U}}^k$ *can be attained by a sequentially perfect FAP, that is,* $\overline{\mathcal{U}}^k \subset \mathcal{E}$.

Since elements of $\overline{\mathcal{U}}^k$ *are quite cumbersome, we report them using floating-point arithmetic with an exponent of base* 3.

- *The set* $\overline{\mathcal{U}}^1$ *is the convex hull of* $(0, 0, 0.068, 0.043, 0.641)$, $(0, 0.241, 0, 0.158, 0.443)$, $(0, 0.635, 0, 0.408, 0)$, $(0, 0.154, 0.505, 0.414, 0)$, *and* $(0, 0, 0.589, 0.365, 0.830)$.

- *The set* $\overline{\mathcal{U}}^2$ *is the convex hull of* $(0, 0, 0.616, 0.417, 0)$, $(0, 0.625, 0, 0.417, 0)$, $(0, 0.236, 0, 0.157, 0.445)$, *and* $(0, 0, 0.067, 0.045, 0.638)$.

Furthermore, the convex combination of $(0, 0, 0.616, 0.417, 0)$ *and* $(0, 0.625, 0, 0.417, 0)$ *with weights* 0.2 *and* 0.8 *is approximately equal to* $\gamma^{t^2}(\iota)$. *So, in this example, the elements of* $\overline{\mathcal{U}}^2$ *can be used to approximate the payoffs along the FAP* ι *that has infinite cycles.*

5 Extensions

We developed the Essential APS approach to FAPs, where only a single player is allowed to quit with a positive rate in every time instance. Yet, there may be equilibria where several players quit with a positive rate simultaneously, or even where players may quit at a given time instance with probability bounded away from 0. To show the versatility of the essential APS approach, in this section, we extend it to absorption paths where several players are allowed to quit with a positive rate simultaneously.

5.1 Generalized FAPs

Recall that under an FAP ι, a single player quits with rate one in each connected component (t, t') of $[0, 1) \setminus H(\iota)$. We now define a generalization of FAPs, allowing several players to quit at the same time over such connected components. These generalized FAPs can be seen as the limit case of strategy profiles in the original game, where the players quit alternately with vanishing probabilities.

Definition 5.1 *A Generalized Flesch Absorption Path (GFAP) is a right-continuous, piece-wise constant map $\alpha : [0, 1) \to \Delta(I)$.*

Let α be a GFAP. For every $0 \le t < 1$ and for each player $i \in I$, the value of $\int_t^1 \alpha_i(s)ds$ represents the probability that the play terminates by the action profile (Q_i, C_{-i}) in the interval $[t, 1)$. Since the probability of absorption in $[t, 1)$ is $1 - t$, the *expected payoff* after absorption probability t is given by

$$\gamma^t(\alpha) := \sum_{i \in I} \frac{\int_t^1 \alpha_i(s)ds}{1 - t} \cdot R_i. \tag{16}$$

The notion of sequential perfectness for GFAPs is analogous to the one for FAPs:

Definition 5.2 *A GFAP α is* sequentially perfect *if, for every $t \in [0, 1)$, $\gamma^t(\alpha) \ge \vec{0}$ and $\gamma_i^t(\alpha) = 0$ whenever $\alpha_i(t) > 0$.*

Let $\underline{\Upsilon}$ be the set of sequentially perfect GFAPs and $\underline{\mathcal{E}}$ be the set of sequentially perfect GFAP payoffs:

$$\underline{\mathcal{E}} := \{w \in \mathbb{R}^I : \exists \alpha \in \underline{\Upsilon} \text{ s.t. } w = \gamma_0(\alpha)\}.$$

We will show how $\underline{\mathcal{E}}$ can be characterized using a variation of the Essential APS algorithm. Since the construction is parallel to that in Section 4, we will skip most of the intermediate steps.

Let us define the directed graph $(\underline{I}, \underline{L})$ we need for the algorithm. The vertices are here the non-empty subsets of players $\underline{I} := 2^I \setminus \{\emptyset\}$, and we add an edge between two sets $N, M \in \underline{I}$ if the players in N can be followed by M as quitters in a sequentially perfect GFAP. With analogous arguments as in Lemma 4.1, this is equivalent to the following: $(N, M) \in \underline{L}$ if and only if $N \ne M$ and there exist $(\lambda_i)_{i \in N} \in (0, 1]^N$ and $(\lambda'_i)_{i \in M} \in (0, 1]^M$ such that

$$\begin{cases} \sum_{i \in N} \lambda_i R_{i,j} \ge 0, & \forall j \in M, \\ \sum_{i \in M} \lambda'_i R_{i,j} \le 0, & \forall j \in N. \end{cases} \tag{17}$$

For each $N \in \underline{I}$, let $\underline{S}_N$ be the set of subsets M with $(N, M) \in \underline{L}$. Let $\mathbb{I}$ be the set of strongly connected components of $(\underline{I}, \underline{L})$ with a typical element $(\underline{N}, \underline{L}_{\underline{N}}) \in \mathbb{I}$. The set of all subsets of players belonging to some strongly connected components reachable from $\underline{N}$, excluding $\underline{N}$, is denoted by $\widehat{\underline{N}}$.

We now define the Essential APS operator similarly to the definition in Section 4. For each strongly connected component $(\underline{N}, \underline{L}_{\underline{N}}) \in \mathbb{I}$, every element $N \in \underline{N}$, and every collection of sets $(E_M)_{M\in\underline{N}\cup\widehat{\underline{N}}} \subseteq (\mathcal{R}_I)^{\underline{N}\cup\widehat{\underline{N}}}$, set

$$\underline{\mathbf{F}}_{N,\underline{N}}\left((E_M)_{M\in\underline{N}}|(E_M)_{M\in\widehat{\underline{N}}}\right) := (\mathcal{R}_N \cap \mathcal{H}_N) \cup \Big\{w \in \mathbb{R}^I :\ \exists(\lambda, v) \in [0,1]^N \times \cup_{M\in\underline{S}_N} E_M \text{ s.t. } w = \sum_{i\in N}\lambda_i R_i + \big(1 - \sum_{i\in N}\lambda_i\big)v, w_j = 0\ \forall j \in N\Big\}.$$

Stack together $\underline{\mathbf{F}}_{N,\underline{N}}$'s as follows:

$$\underline{\mathbf{F}}_{\underline{N}}\left((E_M)_{M\in\underline{N}}|(E_M)_{M\in\widehat{\underline{N}}}\right) := \left(\underline{\mathbf{F}}_{N,\underline{N}}\left((E_M)_{M\in\underline{N}}|(E_M)_{M\in\widehat{\underline{N}}}\right)\right)_{N\in\underline{N}}.$$

Following the same inductive approach as in Section 4, we build the largest invariant sets of this generalized Essential APS operator: for $\underline{N} \in \mathbb{I}$ such that $\widehat{\underline{N}} = \emptyset$, we define

$$\underline{\mathcal{F}}_{\underline{N}} := co\left(\bigcup_{N\in\underline{N}}\left(co(\{R_i : i \in N\}) \cap \mathcal{H}_N\right)\right) \cap \mathbb{R}^I_+,$$

and set

$$(\overline{\underline{\mathcal{E}}}_N)_{N\in\underline{N}} := \bigcap_{n=0}^{\infty}\left(\underline{\mathbf{F}}_{\underline{N}}\right)^{(n)}\left((\underline{\mathcal{F}}_{\underline{N}})^{\underline{N}}|\emptyset\right).$$

Then, for arbitrary $\underline{N} \in \mathbb{I}$ such that the sets $(\overline{\underline{\mathcal{E}}}_N)_{N\in\widehat{\underline{N}}}$ are known, define

$$\underline{\mathcal{F}}_{\underline{N}} := co\left(\bigcup_{N\in\underline{N}}\left(\left(co(\{R_i : i \in N\}) \cup \bigcup_{M\in\underline{S}_N\cap\widehat{\underline{N}}}\overline{\underline{\mathcal{E}}}_M\right) \cap \mathcal{H}_N\right)\right) \cap \mathbb{R}^I_+,$$

and set

$$(\overline{\underline{\mathcal{E}}}_N)_{N\in\underline{N}} := \bigcap_{n=0}^{\infty}\underline{\mathbf{F}}_{\underline{N}}^{(n)}\left((\underline{\mathcal{F}}_{\underline{N}})^{|\underline{N}|}|(\overline{\underline{\mathcal{E}}}_N)_{N\in\widehat{\underline{N}}}\right).$$

The following theorem characterizes the set of payoffs that can be implemented by GFAPs. We omit its proof, because it is parallel to the proof of Theorem 4.10.

Theorem 5.3 *Suppose that for every strongly connected component $\underline{N} \in \mathbb{I}$, we have $\underline{\mathcal{F}}_{\underline{N}} \cap \{w \in \mathbb{R}^I : w_i = 0,\ \forall N \in \underline{M},\ \forall i \in N\} = \emptyset$ for all simple circuits $\underline{M} \subseteq \underline{N}$. Then,*

$$\underline{\mathcal{E}} = \bigcup_{N\in\underline{I}}\overline{\underline{\mathcal{E}}}_N.$$

Furthermore, for each $w \in \underline{\mathcal{E}}$, there exists a sequentially perfect GFAP α with $\gamma_0(\alpha) = w$ such that the ordinality of $H(\alpha)$ is at most the ordinality of $\mathbb{N}$.

5.2 Characterizing continuous equilibrium payoffs

In this last section, we consider the most general framework of continuous quitting. We adapt the APS approach to approximate the whole set of subgame-perfect equilibrium payoffs where in the corresponding ε-equilibria, players quit with infinitesimal probabilities throughout the play.

Definition 5.4 *A continuous absorption path (CAP) α is a measurable map from $[0,1)$ to $\Delta(I)$.*

An expected payoff path under CAPs is still given by Eq. (16). The notion of sequential perfectness for CAPs is analogous to the corresponding notion for GFAPs.

Definition 5.5 *A CAP α is* sequential perfect *if for every $t \in [0,1)$, $\gamma^t(\alpha) \geq \vec{0}$ and $\gamma_i^t(\alpha) = 0$ for a.e. $t \in [0,1)$ such that $\alpha_i(t) > 0$.*

We are interested in the set of payoffs $\underline{\mathcal{E}}^*$ in which every element can be attained by a sequential perfect CAP:

$$\underline{\mathcal{E}}^* = \big\{w \in \mathbb{R}^I : \exists \text{ sequentially perfect CAP } \alpha \text{ s.t. } w = \gamma^0(\alpha)\big\}.$$

We shall show how to characterize the set $\underline{\mathcal{E}}^*$, in a spirit similar to Solan and Vieille (2001, Proposition 2.4) and Simon (2007, Theorem 3). To overcome the problem that CAPs may be not piecewise-constant, we approach the elements of $\underline{\mathcal{E}}^*$ by the expected payoffs of GFAPs on which we impose a uniform lower bound on their quitting rates and which satisfy a certain relaxed notion of sequential perfectness. More precisely, we define for all $\varepsilon > 0$ the operator

$$\begin{aligned}\underline{\mathbf{T}}_\varepsilon(E) := \Big\{w \in \mathbb{R}^I_+ :\ \exists\, (\lambda, v) \in [0,1]^I \times E \text{ s.t. } w = \textstyle\sum_{i\in I} \lambda_i R_i + (1 - \sum_{i\in I} \lambda_i) v,\\ \textstyle\sum_{i\in I} \lambda_i \in [\varepsilon, 1] \text{ and } w_j \leq \varepsilon \|R\| \text{ whenever } \lambda_j > 0\Big\},\end{aligned}$$

where $\|R\| := \max_{i,j} |R_{i,j}|$ is the max norm of R.

Theorem 5.6 $\underline{\mathcal{E}}^* = \bigcap_{\varepsilon>0} \bigcap_{n=0}^\infty \underline{\mathbf{T}}_\varepsilon^{(n)}(\mathcal{R}_I)$.

Proof. First of all, we unpack $\bigcap_{\varepsilon>0} \bigcap_{n=0}^\infty \underline{\mathbf{T}}_\varepsilon^{(n)}(\mathcal{R}_I)$ and characterize its elements more explicitly. For all $\varepsilon > 0$, $w \in \bigcap_{n=0}^\infty \underline{\mathbf{T}}_\varepsilon^{(n)}(\mathcal{R}_I)$ if and only if there exists a sequence $(w^n, \lambda^n)_{n\in\mathbb{N}} \subset (\mathbb{R}^I_+ \times [0,1]^I)^{\mathbb{N}}$ with $w^0 = w$ such that for each $n \in \mathbb{N}$,

$$\begin{cases} w^n = \sum_{i\in I} \lambda_i^n R_i + (1 - \sum_{i\in I} \lambda_i^n) w^{n+1}, \\ \sum_{i\in I} \lambda_i^n \in [\varepsilon, 1] \text{ and } w_i^n \leq \varepsilon \|R\| \text{ whenever } \lambda_i^n > 0. \end{cases} \tag{18}$$

We now show that, for any fixed $\varepsilon > 0$, $\underline{\mathcal{E}}^* \subseteq \bigcap_{n=0}^\infty \underline{\mathbf{T}}_\varepsilon^{(n)}(\mathcal{R}_I)$. Let α be a sequentially perfect CAP and define $(t^n)_{n\in\mathbb{N}}$ as follows: $t^0 := 0$, $t^{n+1} := t^n + \varepsilon(1 - t^n)$ for all $n \in \mathbb{N}$.

Note that $t^n \nearrow 1$ as $n \to \infty$. In addition, define a sequence $(w^n, \lambda^n)_{n\in\mathbb{N}}$ by setting for each $n \in \mathbb{N}$, $w^n := \gamma^{t^n}(\alpha)$, and

$$\lambda_i^n := \frac{\int_{t^n}^{t^{n+1}} \alpha_i(s)ds}{1-t^n}, \quad i \in I.$$

Next, we prove that the sequence $(w^n, \lambda^n)_{n\in\mathbb{N}}$ satisfies the relation (18). Since α is sequentially perfect, $w^n \geq \vec{0}$ for all $n \in \mathbb{N}$. By definition, $\sum_{i\in I} \alpha_i(t) = 1$ for all $t \in [0,1)$, thus $\sum_{i\in I} \lambda_i^n = \varepsilon$ for all $n \in \mathbb{N}$.
In this generalized framework it still holds that, for all $t > t_n$,

$$w^n = \sum_{i\in I} \frac{\int_{t^n}^{t} \alpha_i(s)ds}{1-t} \cdot R_i + \frac{1-t}{1-t^n}\gamma^t(\alpha). \tag{19}$$

Setting $t = t_{n+1}$ in Eq. (19), we recover the relation between w_n and w_{n+1}. Moreover, for some $i \in I$, if $\lambda_i^n > 0$, then there exists $t' \in (t^n, t^{n+1}]$ such that $\gamma_i^{t'}(\alpha) = 0$, because $Leb(\{s \in (t^n, t^{n+1}] : \alpha_s^i > 0\} \neq 0$ and α is sequentially perfect. Eq. (19) for $t = t'$ gives $w_i^n \leq \varepsilon\|R\|$.

The arguments and results used to prove the reverse direction are borrowed from AKRS. Let $w \in \bigcap_{\varepsilon>0}\bigcap_{n=0}^{\infty} \underline{\mathbf{T}}_{\varepsilon}^{(n)}(\mathcal{R}_I)$. For every $\varepsilon > 0$, there exists a sequence $(w^{\varepsilon,n}, \lambda^{\varepsilon,n}) \in (\mathbb{R}_+^I \times [0,1])^{\mathbb{N}}$ that satisfies (18). Let $(t^{\varepsilon,n})_{n\in N}$ be given by $t^{\varepsilon,0} := 0$, and $t^{\varepsilon,n+1} := t^{\varepsilon,n} + (1 - t^{\varepsilon,n})\sum_{i\in I} \lambda_i^{\varepsilon,n}$ for all $n \in \mathbb{N}$. Since $\sum_{i\in I} \lambda_i^{\varepsilon,n} \geq \varepsilon > 0$, the following CAP $\alpha^\varepsilon : [0,1) \to \Delta(I)$ is well-defined:

$$\alpha_i^\varepsilon(t) = \frac{\lambda_i^{\varepsilon,n}}{\sum_{j\in I} \lambda_j^{\varepsilon,n}} \ \text{ if } \ t \in [t^{\varepsilon,n}, t^{\varepsilon,n+1}) \ \text{ for } i \in I. \tag{20}$$

Define $\pi^\varepsilon : [0,1) \times A^* \times \mathbb{R}$ as follows: $\pi_t^\varepsilon(Q_i, C_{-i}) = \int_0^t \alpha_i^\varepsilon(s)ds$ and $\pi_t^\varepsilon(a) = 0$ for all $a \in A^*_{\geq 2}$. The function π^* is an absorption path as defined in AKRS. Since, by Proposition 4.11 of AKRS, the space of absorption paths is sequentially compact, the sequence $(\pi^\varepsilon)_{\varepsilon>0}$ admits a convergent subsequence. Let π be the associated limit point. For each $t \in [0,1)$, since $\widehat{\pi}_t^\varepsilon := \sum_{i\in I} \pi_t(Q_i, C_{-i}) = t$ for all ε, we also have $\widehat{\pi}_t = t$, and it follows that $\alpha_t := (\dot{\pi}_t(Q_i, C_{-i}))_{i\in I}$ defines a CAP. As shown in AKRS, the expected payoff is a continuous function of absorption paths, thus $\gamma(\alpha) = w$. Finally, α is sequentially perfect, because the sequence $(w_{\varepsilon,n}, \lambda_{\varepsilon,n})$ satisfies (18) for each $\varepsilon > 0$. Taking all of the pieces together, we conclude $w \in \underline{\mathcal{E}}^*$. ■

6 Conclusion

The APS approach is an iterative method to characterize a set of equilibrium payoffs (and strategies that attain them) in discounted games.

In this paper, we adapted the APS approach to study undiscounted subgame-perfect equilibria in quitting games, and characterized a class of subgame-perfect equilibrium payoffs and the corresponding equilibrium strategy profiles in this class of games. It is interesting to know whether the approach can be extended to find *all* subgame perfect equilibrium payoffs in quitting games, and not only those that are supported by FAPs, GFAPs, or CAPs.

Since quitting games are both stopping games and stochastic games, it is also interesting to know whether our approach can be extended to more general classes of stopping games and stochastic games. One challenge in extending it to general stochastic games is that while in quitting games the payoff is obtained when the game terminates, in undiscounted stochastic games the total payoff is the long-run average of stage payoffs, and it is not clear how to adapt our approach to this setup.

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