On Dynamic Pricing

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Motivation

- There are many reasons firms can price discriminate on the basis of **timing of purchases**.
- Focus here on **sequential arrival of private info**.

> *"In this case the monopolist faces the problem of structuring his pricing so that consumers 'self-select' into appropriate categories."*
>
> — Varian [1989]

- Applications:
  - **unit** demand: refundable and non-refundable airfares.
  - **repeated** demand: options and forwards on electricity.

In this paper

- Build a model of dynamic price discrimination:
  - single seller.
  - one buyer who's value for trade can change at random times.
- Look at two settings:
  - unit demand.
  - repeated demand.
- Construct a simple dynamic pricing instrument:

American option on forwards (AOF)

Results

- In the unit demand setting:
  - Characterize the best AOF.
  - Show that the best AOF **implements the dynamic optimum**.
  - Build an alternative implementation with refunds.
- In the repeated demand setting:
  - Construct a pricing strategy with AOFs.
  - Show that the pricing strategy is **approximately optimal**.
- Technical insight: deal with **global incentive constraints**.

Plan

- Unit demand:
  - Example.
  - Continuous time model.
  - Refund contracts.
- Repeated demand:
  - Dynamics of sales.
  - Approximate optimality.

Unit demand

Example

- Two periods: today and tomorrow.
- The seller has one unit to sell tomorrow.
- Today: buyer's value is $v_1 \sim U[0,1]$.
- Tomorrow: buyer's value v_2 might be different from v_1:
  - wp $1/2$, the value is unchanged, ie $v_2 = v_1$.
  - wp $1/2$, the value is redrawn, ie $v_1 \perp v_2$ with $v_2 \sim U[0,1]$.
- Preferences are linear, production is costless.
- The seller wants to max her profits.

Example: spot pricing

- Suppose that the seller offers the good for a spot price α.
- The buyer buys the good if and only if $v_2 \geqslant \alpha$.
- Seller's expected profit is

$$\alpha \mathbb{P}(v_2 \geqslant \alpha) = \alpha(1 - \alpha)$$

- The optimal price is $\alpha = 1/2$ which yields $1/4$.

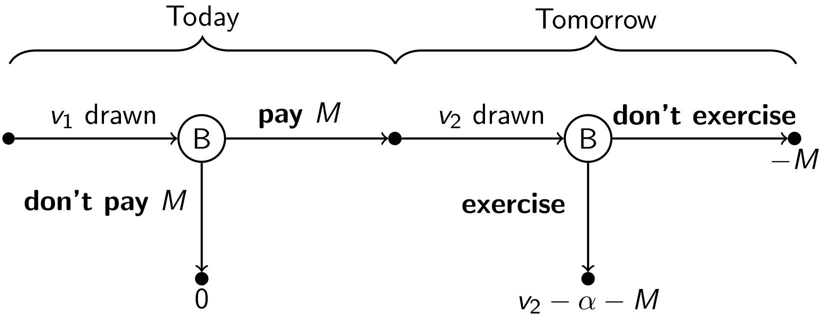
Example: European option.

- Buyer's expected payoff from the spot contract:

$$\mathbb{E}\{(v_2-\alpha)^+|v_1\} = 1/2\times(v_1-1/2)^+ + 1/2\times\underbrace{\mathbb{E}\{(v_2-1/2)^+\}}_{=1/8} > 0$$

- The seller leaves too much surplus on the table.
- Consider European option $\{\alpha, M\}$ where
  - α is a spot price.
  - M is a **flat upfront payment** (option premium).

Example: buyer's problem



Example: optimal European option

- What is the optimal combination of α and M?
- Choose the largest M st the buyer **always takes the option**:

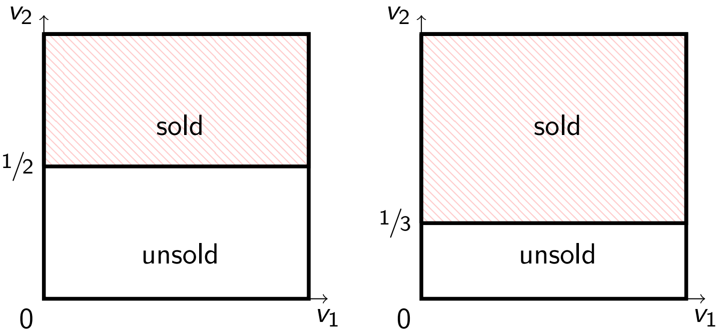
$$M(\alpha) = \mathbb{E}\left\{\left(v_2 - \alpha\right)^+ | v_1 = 0\right\}$$

- Seller's profit is then

$$\underbrace{M(\alpha)}_{\text{premium}} + \underbrace{\alpha(1-\alpha)}_{\text{spot pricing}}$$

- The optimal offer is $\alpha = 1/3$, $M = 1/9$ which yields $1/3$.
- **Can not be improved** by choosing larger M.

Example: spot pricing vs European option

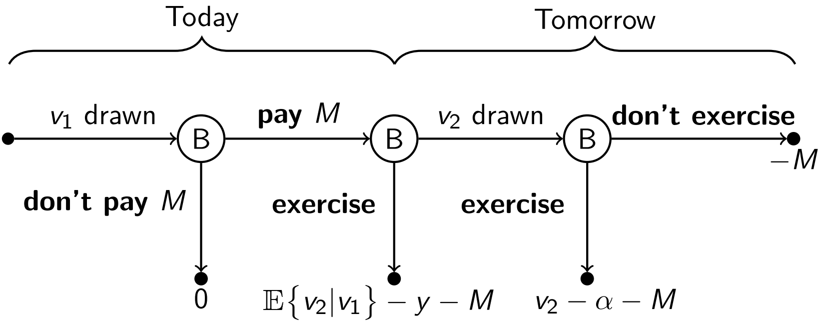


(a) spot pricing (b) European option

Example: American option on forwards

- Can the seller sort the buyer today and increase her profit?
- Consider American option $\{\alpha, M, y\}$ where
  - α is a spot (strike) price.
  - M is a flat upfront fee (option premium).
  - y is a **forward price**.

Example: buyer's problem

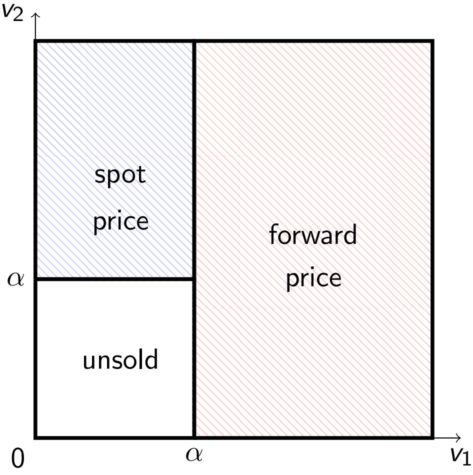


Example: screening by AOF

- As before, let $M(\alpha)$ st the buyer **always takes the option**.
- Choose y st the **buyer self-selects into 2 groups** today:
  - $v_1 \geqslant$ **threshold**: exercise the forward today.
  - $v_1 <$ **threshold**: wait for the spot price tomorrow.
- Tomorrow, the buyer will buy iff $v_2 \geqslant \alpha$.
- Set **threshold = spot price**, y solves:

$$\underbrace{\mathbb{E}\{v_2|v_1=\alpha\} - y(\alpha)}_{\text{net value of taking the forward for } v_1=\alpha} \quad = \quad \underbrace{\mathbb{E}\{(v_2-\alpha)^+|v_1=\alpha\}}_{\text{net value of waiting for } v_1=\alpha}$$

Example: screening by AOF



Example: optimal AOF

- Seller's profit can be written as

$$\underbrace{M(\alpha)}_{\text{premium}} + \underbrace{\alpha(1-\alpha)}_{\text{spot pricing}} + \underbrace{1/2 \times (1-\alpha)\int_0^{\alpha} v dv}_{\text{extra due to forward pricing}}$$

- The optimal AOF is

$$\{\alpha = 7/18, M = 121/1296, y = 455/1296\}$$

which yields approximately 0.387.

Example: direct mechanisms

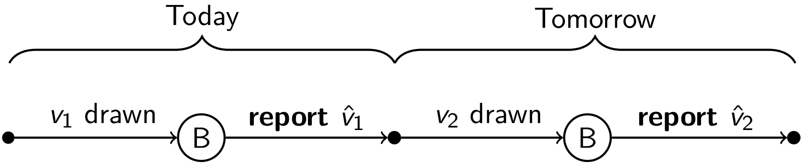
- Will show that the best AOF implements the optimum.
- Invoking revelation principle, the seller offers a contract

$$\left\{q(\hat{\mathbf{v}}^{\mathbf{2}}), p(\hat{\mathbf{v}}^{\mathbf{2}})\right\}$$

where $\hat{\mathbf{v}}^{\mathbf{2}} = (\hat{v}_1, \hat{v}_2)$ is a history of reports, and

  - $q(\hat{\mathbf{v}}^{\mathbf{2}}) \in \{0, 1\}$ is an allocation.
  - $p(\hat{\mathbf{v}}^{\mathbf{2}}) \in \mathbb{R}$ is a payment.

Example: buyer's problem



Buyer's payoff $= v_2 q(\hat{\mathbf{v}}^2) - p(\hat{\mathbf{v}}^2)$

- **IC tomorrow**: $\hat{v}_2 = v_2$ is optimal $\forall\ \mathbf{v}^2$ given $\hat{v}_1 = v_1$.
- Note IC tomorrow $\implies$ truthtelling tomorrow even if $\hat{v}_1 \neq v_1$.
- **IC today**: $\hat{v}_1 = v_1$ is optimal given truthtelling tomorrow $\forall\ v_1$.

Example: IC tomorrow

- IC tomorrow can be written as

$$u(\mathbf{v^2}) := v_2 q(\mathbf{v^2}) - p(\mathbf{v^2}) \geqslant v_2 q(v_1, \hat{v}_2) - p(v_1, \hat{v}_2)$$

- Note $u(v_1, .)$ must be convex as the max of linear functions.
- By the Envelope theorem:

$$\frac{\partial}{\partial v_2} u(\mathbf{v^2}) = q(\mathbf{v^2})$$

- Also, can show that

convexity + envelope = IC tomorrow

Example: IC today

- IC tomorrow can be written as

$$U(v_1) := \mathbb{E}\{u(\mathbf{v}^2)|v_1\} = {}^1\!/\!_2 \times u(v_1, v_1) + {}^1\!/\!_2 \times \int_0^1 u(v_1, v_2)dv_2$$
$$\geqslant {}^1\!/\!_2 \times u(\hat{v}_1, v_1) + {}^1\!/\!_2 \times \int_0^1 u(\hat{v}_1, v_2)dv_2$$

- Note U must be convex as the max of convex functions.
- By the Envelope theorem:

$$U'(v_1) = {}^1\!/\!_2 \times \frac{\partial}{\partial v_2} u(v_1, v_2)\Big|_{v_2 = v_1} \overset{\text{IC tomorrow}}{=} {}^1\!/\!_2 \times q(v_1, v_1)$$

Example: integral monotonicity

- Unfortunately,

$$\textbf{convexity} + \textbf{envelope} \neq \textbf{IC today}$$

- Subtract $U(\hat{v}_1)$ from the both sides of IC today:

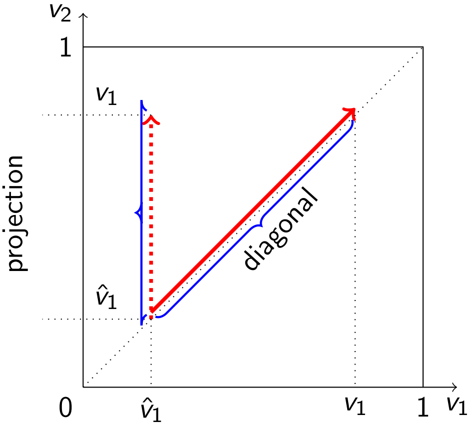
$$U(v_1) - U(\hat{v}_1) \geqslant {}^1\!/_2 \times \left[u(\hat{v}_1, v_1) - u(\hat{v}_1, \hat{v}_1) \right]$$

- Equivalently, in the integral form:

$$\int_{\hat{v}_1}^{v_1} \underbrace{U'(v)}_{\substack{= {}^1\!/_2 \times q(v,v) \\ \text{envelope today}}} dv \geqslant \int_{\hat{v}_1}^{v_1} \underbrace{{}^1\!/_2 \times \frac{\partial}{\partial v_2} u(\hat{v}_1, v)}_{\substack{= {}^1\!/_2 \times q(\hat{v}_1, v) \\ \text{envelope tomorrow}}} dv$$

Example: integral monotonicity

$$\int_{\hat{v}_1}^{v_1} q(v, v)dv \geqslant \int_{\hat{v}_1}^{v_1} q(\hat{v}_1, v)dv$$



Example: seller's profit

- By linearity of preferences:

$$\textbf{profit} = \textbf{surplus} - \textbf{buyer's payoff}$$

- **surplus** $= \mathbb{E}\{v_2 q(\mathbf{v^2})\}$ is given by

$$1/2 \times \int_0^1 v_1 q(v_1, v_1) dv_1 + 1/2 \times \int_{[0,1]^2} v_2 q(\mathbf{v^2}) d\mathbf{v^2}$$

- Using integration by parts, **buyer's payoff** $= \mathbb{E}\left[U(v_1)\right]$ is

$$\underbrace{U(0)}_{=0 \text{ by IR}} + \int_0^1 (1 - v_1) \underbrace{U'(v_1)}_{\substack{=1/2 \times q(v_1, v_1) \\ \text{envelope today}}} dv_1$$

Example: standard approach (FOA)

- The common approach is the first-order approach (FOA) ignores the constraints:
  - Integral monotonicity.
  - Convexity tomorrow, ie $q(v_1, .)$ is $\uparrow$.
- Then, we maximize profit pointwise and obtain:

$$q_2(\mathbf{v^2}) = \mathbb{1}\{v_1 = v_2 \geqslant 1/2 \wedge v_1 \neq v_2\}$$

Example: FOA fails

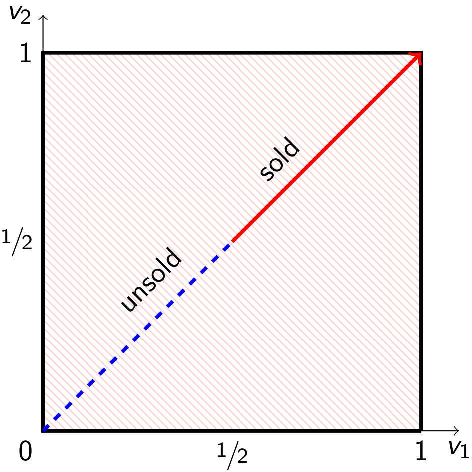


Figure: blue- $q_2(\mathbf{v^2}) = 0$, red- $q_2(\mathbf{v^2}) = 1$.

Example: FOA needs extra info

- The FOA allocation can be implemented iff

  event $v_2 \neq v_1$ is **publicly observed**

- Implementation in the hypothetical setting:
  - Offer the good for $1/2$ whenever $v_1 = v_2$
  - Give the good for free whenever $v_1 \neq v_2$
  - Charge upfront $1/2 \times \mathbb{E}\{v_2 | v_1 \neq v_2\} = 1/2 \times 1/2$.
- In the original setting, the buyer will always claim $\hat{v}_1 \neq \hat{v}_2$.
- Then, trade happens wp one and seller's profit is $1/4 < 0.387$.

Example: optimum

- Seller's problem is to maximize her profit subject to
  - Integral monotonicity.
  - Convexity tomorrow, ie $q(v_1, .)$ is $\uparrow$.

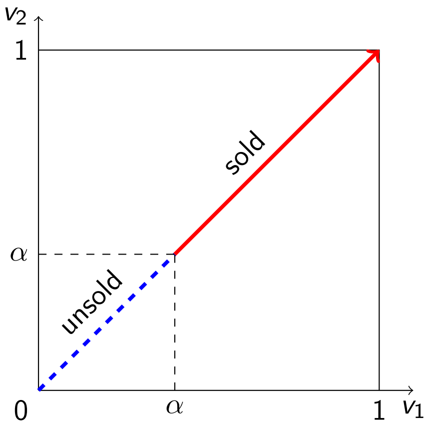
Proposition

Suppose $v_1 \sim U(0,1)$, $v_2 = v_1$ wp $1/2$ or v_2 is independently drawn from $U(0,1)$. The optimal allocation q^* is given by

$$q^*(\mathbf{v}^2) = \mathbb{1}\left\{ \max\{v_1, v_2\} \geqslant 7/18 \right\}$$

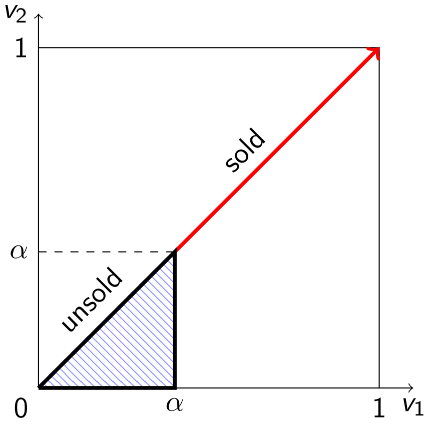
Example: sketch of the proof

U is convex, $U'(v_1) = {}^1\!/_2 \times q(v_1, v_1) \implies \exists \alpha : q(v_1, v_1) = \mathbb{1}\{v_1 \geqslant \alpha\}$



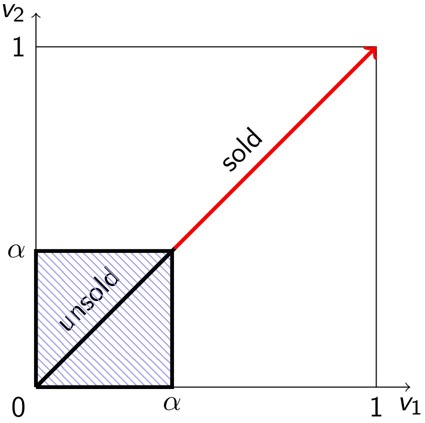
Example: sketch of the proof

$$q(v_1, .) \text{ is } \uparrow \Longrightarrow q(\mathbf{v}^2) \leqslant q(v_1, v_1) = 0 \quad \forall v_2 \leqslant v_1 \leqslant \alpha$$



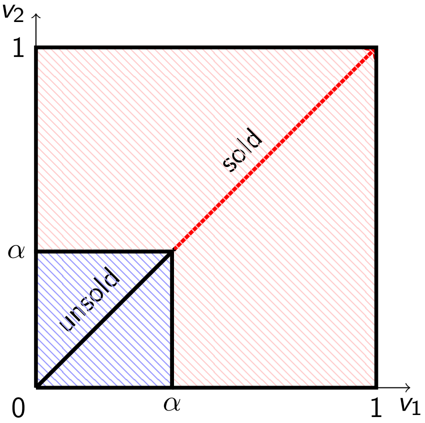
Example: sketch of the proof

$$\text{IM} \implies \int_{v_1}^{v_2} q(v_1, v)dv \leqslant \int_{v_1}^{v_2} q(v, v)dv = 0 \quad \forall v_1 \leqslant v_2 \leqslant \alpha$$

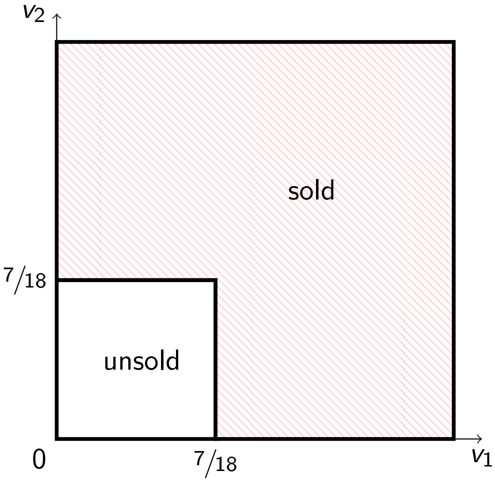


Example: sketch of the proof

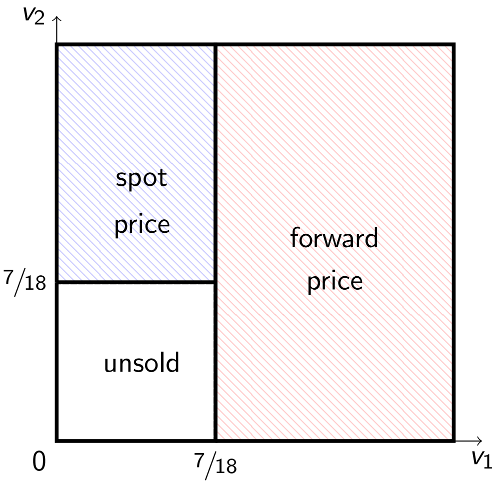
profit is the highest for $q(\mathbf{v}^2) = \mathbb{1}\{\max\{v_1, v_2\} \geqslant \alpha\}$



Example: summary



Example: summary



Related literature

- Screening/price discrimination with fixed population:
  Stokey [1979], Baron & Besanko [1984], **Courty & Li [2000]**, Battaglini [2005], **Eso & Szentes [2007]**, Boleslavsky & Said [2013], Deb [2014], Bergemann & Strack [2015]
- Screening/price discrimination with changing population:
  Conlisk, Gerstner & Sobel [1984], Pesendorfer [2002], Board[2008], Gershkov, Moldovanu & Strack [2014], Garrett [2016], Board & Skrzypacz[2016]
- Theory of dynamic mechanism design:
  Pavan, Segal & Toikka [2014], Eso & Szentes [2017], **Battaglini & Lamba [2018]**, Garrett, Pavan, & Toikka [2018]

Model for the unit demand

- Time is continuous, $t \in [0, T]$.
- The seller has one unit to sell at T.
- Buyer's value $\{V_t\}$ follows renewal process: $V_t = X_{N_t}$ where
  - $\{N_t\}$ is the Poisson process with intensity $\lambda > 0$.
  - $\{X_n\}_{n \geqslant 0}$ is a sequence of independent random draws from F.

American option on forwards

- American option on forwards is defined by

$$\{\alpha, M, \mathbf{y}\} = \left\{\alpha, M, \{y_t\}_{t \le T}\right\}$$

  - α is a spot price.
  - M is an flat upfront fee (option premium).
  - y_t is a forward price at $t \leqslant T$, $y_T = \alpha$.

AOF as an optimal stopping.

- After paying M buyer faces a stopping problem:
  - Stop (exercise) at t: buyer's payoff is $\mathbb{E}\{V_T | V_t\} - y_t - M$.
  - Never: buyer's payoff is $-M$.
- Need to select the AOF and the optimal stopping time.

Optimal AOF

Proposition

There exists α_T^* *such that the optimal AOF is* $\{\alpha_T, M_T, \mathbf{y_T}\}$ *where*

$$M_T = \mathbb{E}\{\left(V_T - \alpha_T\right)^+ | V_0 = 0\}$$

$$y_{t,T} = \alpha_T - \underbrace{\left[1 - e^{-\lambda(T-t)}\right] \int_0^{\alpha_T} F(v)dv}_{\text{forward discount}}$$

In this AOF:

- *The buyer* **always pays** M_T.
- *The buyer* **exercises the option at time** t **iff** $V_t \geqslant \alpha_T$.

Max-contract

- A direct mechanism is defined by

$$\{Q, P\} = \{Q(\hat{\mathbf{V}}^{\mathbf{T}}), P(\hat{\mathbf{V}}^{\mathbf{T}})\}$$

Corollary

The optimal AOF implements the following mechanism:

$$Q^*(\mathbf{V}^{\mathbf{T}}) = 1 \quad \textit{iff} \quad \max_{t \leqslant T} V_t \geqslant \alpha_T$$

$$P^*(\mathbf{V}^{\mathbf{T}}) = M_T + \textit{price at the exercise date}(\mathbf{V}^{\mathbf{T}})$$

- We will refer to $\{Q^*, P^*\}$ as the max-contract.

Optimality of the max-contract

- Formulate the problem using dynamic direct mechanisms.
- Seller's problem is to find the optimal mechanism subject to
  - Incentive compatibility.
  - Individual rationality.

Proposition

The max-contract is the seller's optimal contract.

Gains from randomization

- Allow for randomization: q or $Q \in [0, 1]$.
- The seller can in general do better.

Proposition

1. *For two periods, there is no gain from randomization.*
2. *For more than three periods and continuous time, there are gains from randomization.*

Alternative implementation: refunds

- Non-refundable sales (simple futures):

$$x_t^{non} = \mathbb{E}\{V_T | V_t = \alpha_T\}$$

- Refundable sales (callable futures):

$$x_t^{ref} = \mathbb{E}\{\max\{V_T, \alpha_T\} | V_t = \alpha_T\}$$

- The buyer takes the callable futures at $t = 0$ for x_0^{ref}.
- The first time t when $V_t \geqslant \alpha_T$:
  - The buyer refunds x_t^{ref}.
  - He pays x_t^{non} for the simple futures.

Alternative implementation: refunds

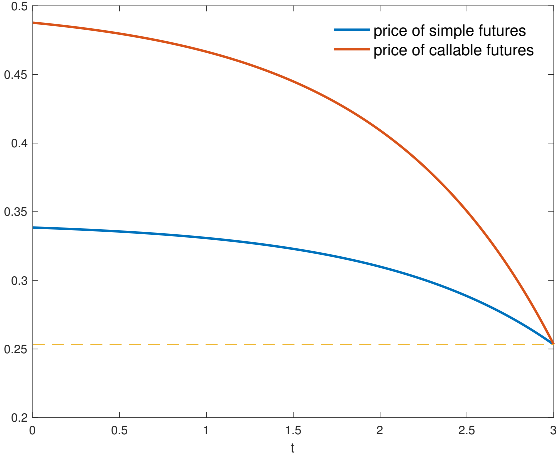


Figure: $F(v) = v$, $\lambda = 1$ and $T = 3$ with $\alpha_T \approx 0.2532$.

Repeated demand

Model for the repeated demand

- Suppose that the seller has one unit to sell at each date.
- The good is perishable.
- The common discount rate is r.
- Signals in the previous model are now payoff relevant values.

AOF/Max-contract for repeated sales

- Paste the previous contracts together
  - Each t-good is sold by the means of the best t-AOF:

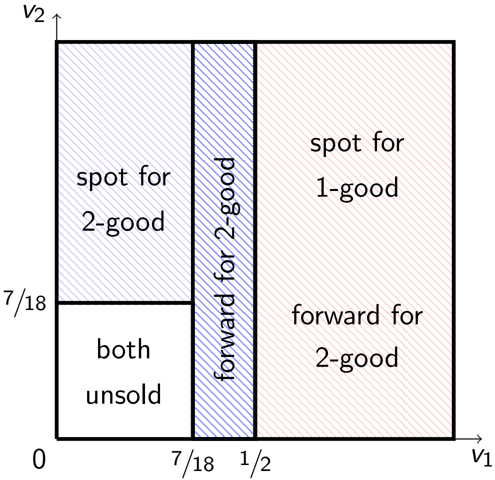
$$\left\{\alpha_t, M_t, \mathbf{y}_t\right\}$$

  - Direct max-contract:

$$Q_t^*(\mathbf{V^t}) = 1 \quad \text{iff} \quad \max_{s \leqslant t} V_s \geqslant \alpha_t$$

$$P_t^*(\mathbf{V^t}) = e^{-rt}\left[M_t + \text{price at the exercise date}(\mathbf{V^t})\right] dT$$

Two period example revisited



Dynamics of sales

Proposition

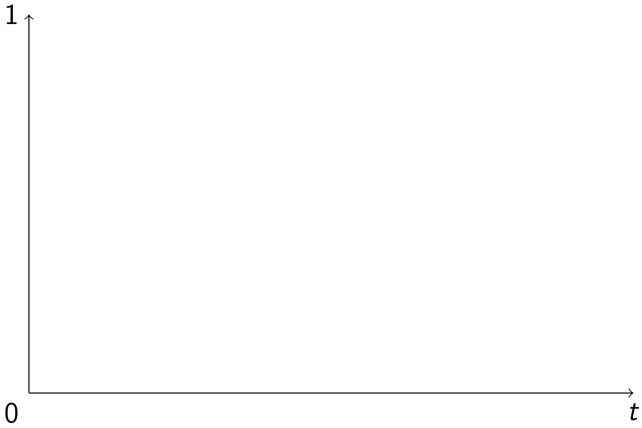
The threshold function $t \mapsto \alpha_t$ is continuous:

- α_0 *equals to the optimal static fixed price.*
- *It is strictly decreasing with* $\lim_{t \to \infty} \alpha_t = 0$.

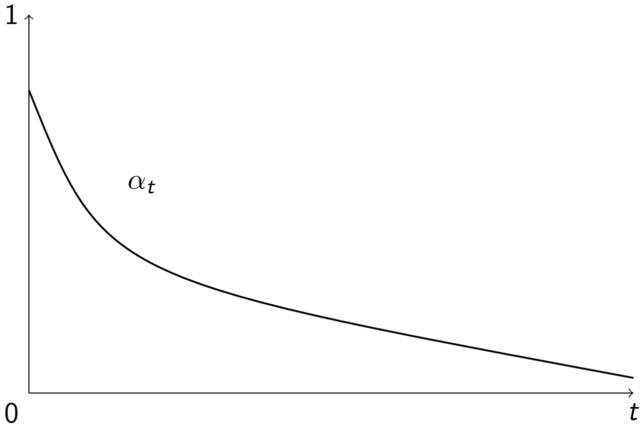
- To describe dynamics, define

$$\textbf{stock, } S_t = \text{ measure of available goods at } t$$

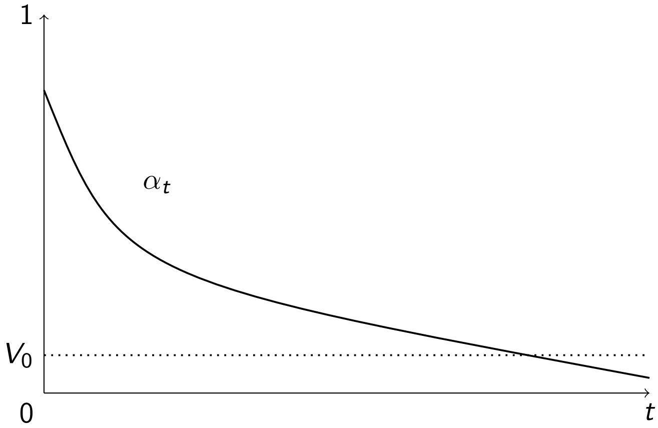
Dynamics of sales



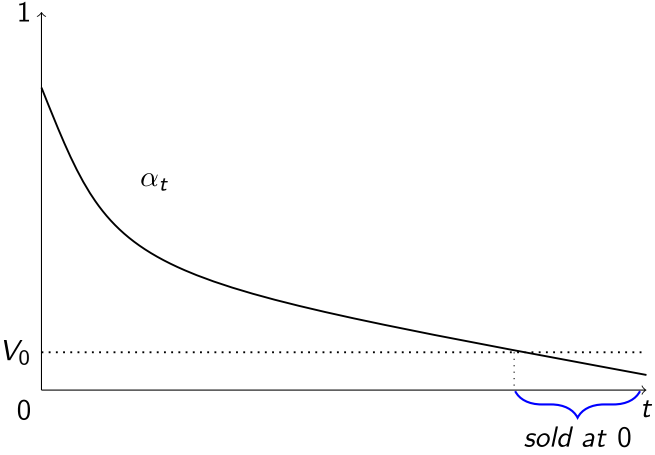
Dynamics of sales



Dynamics of sales



Dynamics of sales



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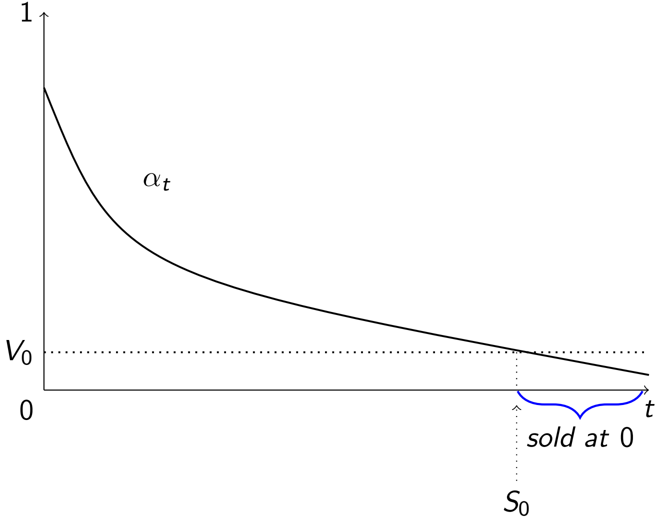

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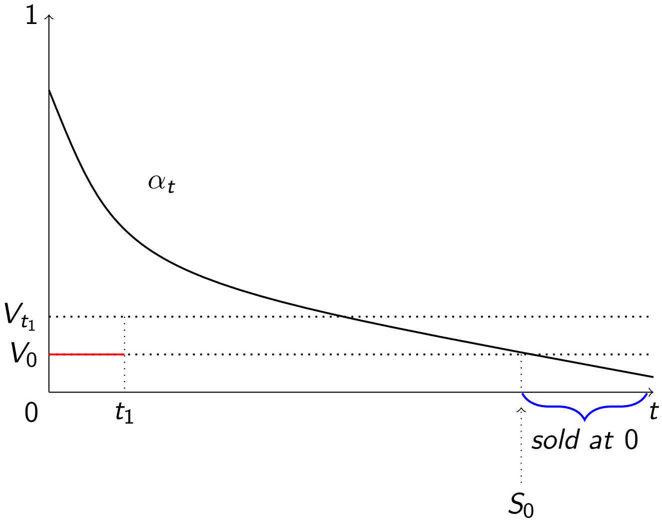

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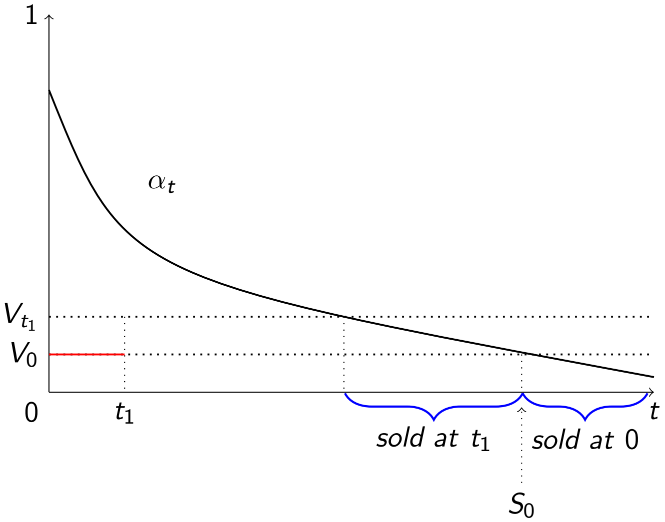

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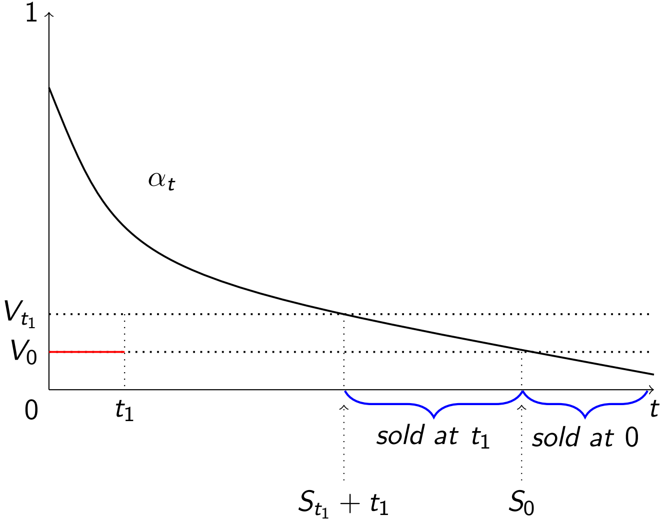

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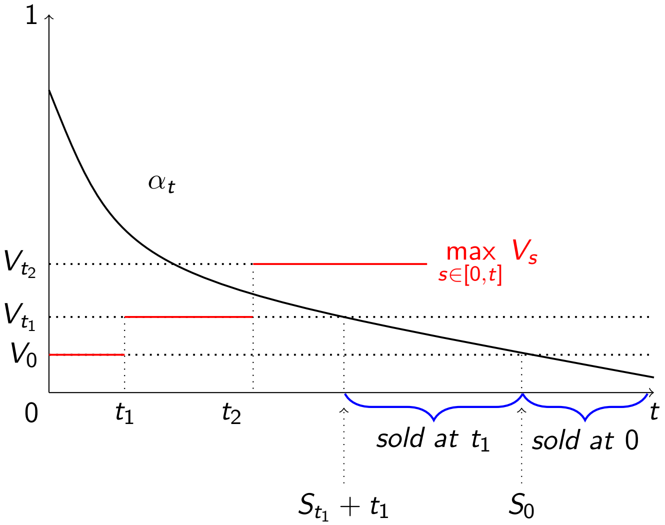
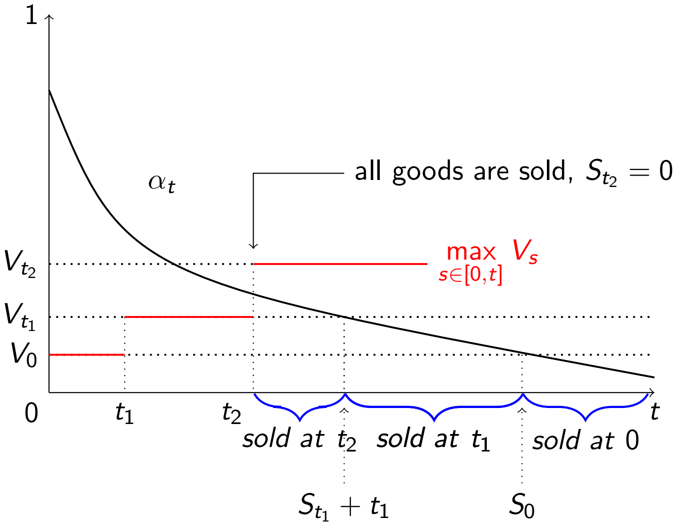
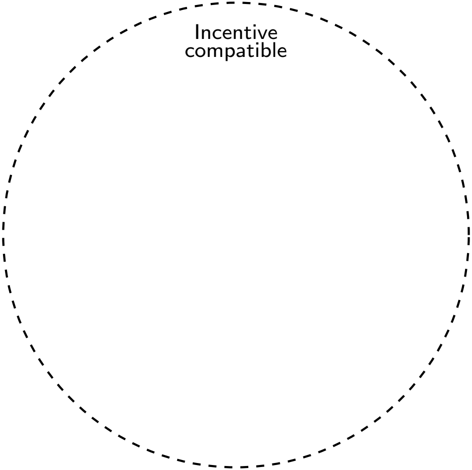


Dynamics of sales



Max-contract is approximately optimal



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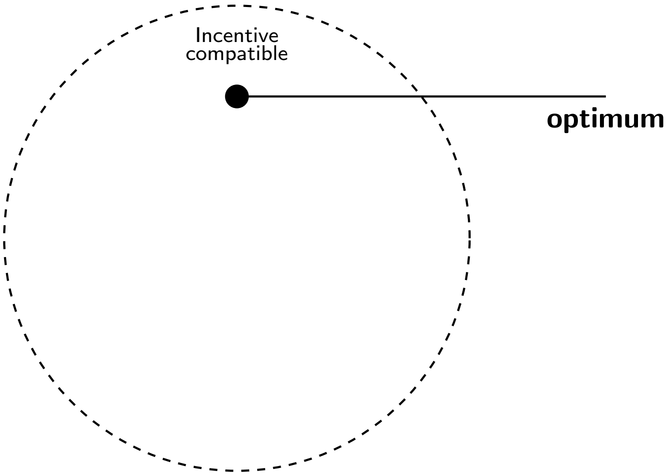
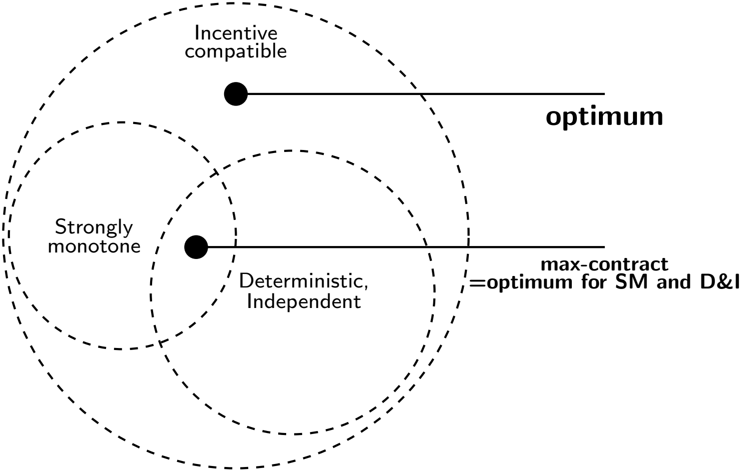
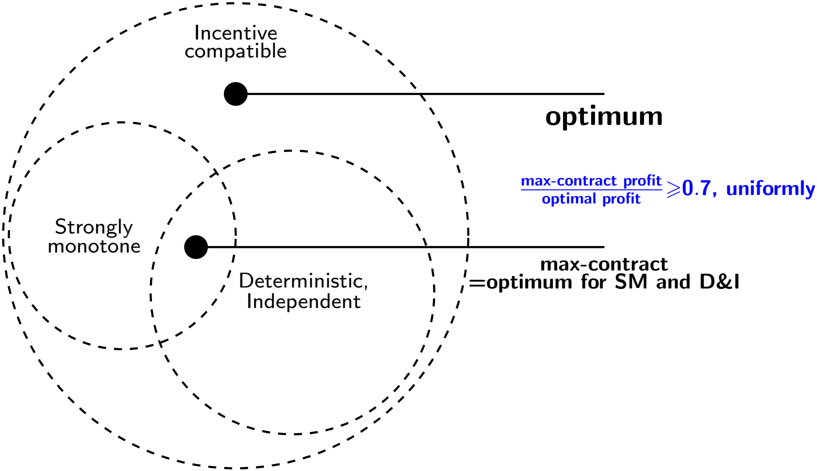


Max-contract is approximately optimal



Max-contract is approximately optimal



AOF/max vs FOA and the optimum

property	FOA	**AOF/max**	optimum
Deterministic	✓	✓	only in continuous time
Independence	✓	✓	only for two and three periods
Implementable (IC)	never for renewal processes	✓	✓

Proposition

The max-contract is the best IC contract in the class of deterministic, independent mechanisms.

Characterization of AOF/Max-contract

- A mechanism $\{\mathbf{Q}, \mathbf{P}\}$ is **strongly monotone** if

$$\mathbf{V}^t \geqslant \hat{\mathbf{V}}^t \quad \Longrightarrow \quad Q_t(\mathbf{V}^t) \geqslant Q_t(\hat{\mathbf{V}}^t)$$

Proposition

The max-contract achieves the optimum within strongly monotone mechanisms.

Profit guarantee

- $\mathcal{R}^{fw}$ = profit from the max-contract.
- $\mathcal{R}^{*}$ = profit from the optimum.
- $\overline{\mathcal{R}}$ = profit from the FOA, ie arrivals are public.

Proposition

$$1 \geqslant \frac{\mathcal{R}^{fw}}{\mathcal{R}^{*}} \geqslant \frac{\mathcal{R}^{fw}}{\overline{\mathcal{R}}} \geqslant \mathcal{L}$$

where $\mathcal{L}$ is a function of r/λ and provides the uniform lower bound across all F st $\lim_{r/\lambda \to 0} \mathcal{L} = \lim_{r/\lambda \to \infty} \mathcal{L} = 1$.

bound on loss from optioning forwards

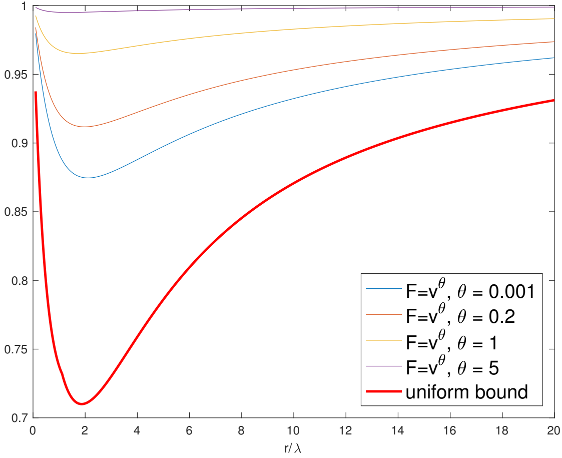


Figure: $\mathcal{R}^{fw}/\overline{\mathcal{R}}$ and $\mathcal{L}$ as functions of r/λ.

Improvement upon AOF with buybacks

- Consider the model in discrete time indexed by $k = 1, 2, 3, \ldots$.
- Suppose at $k = 1$, good 4 bought, but good 3 not.
- For $v_2 \approx 0$ is small, the seller wants $\downarrow$ the spot price α_3^{fw}.
- Can obtain this info by buying back good 4 at $k = 2$.
- The buyer returns good 4 iff $v_2 \approx 0 \implies$ the buyback is a costly signal about v_2, still

  benefits of having better price $>$ costs of buyback.

Final remarks

- This paper:
  - Provides a theory of dynamic pricing with forwards/refunds.
  - Shows that the simple and practical pricing strategy is (approximately) optimal.
- Going forward:
  - Characterize buybacks.
  - Multiple buyers: repeated auctions.
  - Explore more general stochastic processes.
  - Stochastic arrival of buyers.
  - Exogenous restrictions on number of arrivals.